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Comparison between Analytical and Finite Element Solution of Natural Frequencies of Open Circular Cylindrical Roof Shells

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Abstract
An algorithm is derived to compute the natural frequencies of a simply-supported open circular cylindrical roof shell based on the Goldenveizer-Novozhilov shell theory. The algorithm is implemented as a computer program, and the results obtained are verified using the results that have been previously reported. The natural frequencies computed using the analytical method is used to verify the accuracy of solutions obtained from finite element analysis using ABAQUS, ANSYS and SAP2000, using different element formulations and mesh sizes. It is observed that the the natural frequencies computed using the finite element programs are mostly in close agreement with the analytical solution at lower modes, but tend to drift away with increasing mode number, and this can be corrected by using a finer mesh and higher order elements. It is also observed that using a coarse mesh can result in severely inaccurate estimates of the free vibration frequencies. The four-noded linear shell elements were found to have large errors in computed natural frequencies in comparison to quadratic elements, and therefore the use of such elements is not recommended for the dynamic analysis of roof shells.

Keywords
Free Vibration of Shells, Classical Shell Theory, Finite Element Analysis

1. Introduction

Roof shells have shown potential to resist extreme loading during earthquakes. Shell roofs covering large spans have rarely been found to have suffered damage due to earthquakes. The high structural efficiency of shells allow them to be very thin, and as the dynamic forces due to earthquakes is proportional to the mass, the forces acting on shell structures is also smaller in comparison to other structural systems. The high stiffness of shell roofs also means that the natural frequencies are higher than the frequency content of most earthquakes [1, 2]. Good performance of thin concrete shells under earthquake loading, along with their efficient use of possibly costly materials, makes them an economic option for developing countries in regions with high seismic activity.

In order to accurately perform the dynamic analysis of roof shells under earthquake loading, it is necessary to determine the natural frequencies and mode shapes of the shell. The natural free vibration frequencies and mode shapes provide a good insight on the behavior of the structure under dynamic loads. Analysis for earthquake loading is often carried out using the modal response spectrum method, which is popular because of its computational efficiency and is recommended for use by most seismic codes around the world[3, 4]. Researchers have shown that a large number of free vibration modes need to be considered in order to meet the minimum requirement of 90% mass participation found in most seismic codes[5].

The finite element method is currently the preferred method for the dynamic analysis of roof shells. The results obtained from finite element analysis tends to the analytical solution as the element size is reduced. However, the use of finite element method for roof shells should be based upon a sound understanding of the shell theories and the analysis methods being used, and can lead to highly inaccurate results if not used carefully. Comparing the finite element solutions with the solutions from classical shell theories for simple geometries can help increase confidence in the use of finite element programs, while helping to identify potential sources of error.

2. Analytical Modeling of Roof Shell

2.1 Derivation of Equations of Motion

The calculation of natural frequencies and mode shapes of an open cylindrical roof shell is carried out using a semi-analytical method based on the works by Leissa [6]. The Goldenveizer-Novozhilov shell theory, which is a moment theory applicable to both shallow and deep shells has been used as the governing shell theory for the development of the equations of motion. This shell theory can be derived directly by substituting the compatibility and constitutive relationships into the equilibrium equations for an open cylindrical shell, and then using Novozhilov's assumption that $M_{xy} = M_{yx} = H$ and $S = N_{xy} - H/R_2 = N_{yx} - H/R_1$[7].

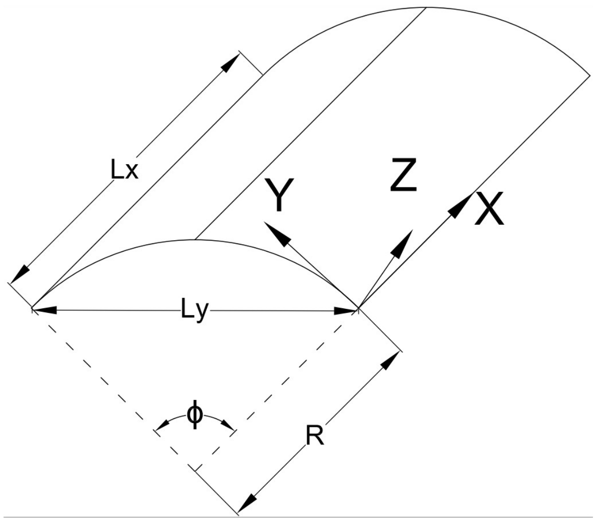

Figure 1: Shell Geometry

The equations of motion were derived for an open cylindrical shell as shown in Fig. 1, based on the method proposed by Leissa [6], by substituting the body forces with inertia terms. The natural frequencies and mode shapes are extracted by substituting displacement functions in the form of a double trigonometric series, as suggested by Leissa[6] and Ostovari-Dailamani[8].

$$
\begin{aligned}
u(x,y,t) &= \sum_i \sum_j A\, cos(\frac{j\pi x}{L})\, sin(\frac{i\pi y}{R\phi}) sin(\omega_{ij}t + \theta_{ij}) \\
v(x,y,t) &= \sum_i \sum_j B\, sin(\frac{j\pi x}{L})\, cos(\frac{i\pi y}{R\phi}) sin(\omega_{ij}t + \theta_{ij}) \\
w(x,y,t) &= \sum_i \sum_j C\, sin(\frac{j\pi x}{L})\, sin(\frac{i\pi y}{R\phi}) sin(\omega_{ij}t + \theta_{ij})
\end{aligned} \tag{1}
$$

Here, A, B and C are the mode shape amplitudes and ω_{ij} is the natural frequency. The integers i and j are the number of half waves along the circumferential (Y) and longitudinal (X) directions respectively. The displacement functions satisfies the boundary condition of a simply-supported cylindrical roof shell,

$$
\begin{aligned}
N_x = v = w = M_x = 0 \;\; along \;\; x = 0, L_x \\
N_y = u = w = M_y = 0 \;\; along \;\; y = 0, R\phi
\end{aligned} \tag{2}
$$

Using the displacement functions (1) for mode (i,j), allows the free vibration equation to be represented as the following symmetric standard eigenvalue problem.

$$
\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} . \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \Omega_{ij} \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} \tag{3}
$$

The elements of the matrix [K] are given by

$$
\begin{aligned}
k_{11} &= \lambda^2 + (\frac{1-\nu}{2})n^2 \\
k_{12} = k_{21} &= \frac{1+\nu}{2} n\lambda \\
k_{13} = k_{31} &= -\nu\lambda \\
k_{22} &= n^2 + (\frac{1-\nu}{2})\lambda^2 + k\{2(1-\nu)\lambda^2 + n^2\} \\
k_{23} = k_{32} &= -n - k\{(2-\nu)\lambda^2 n + n^3\} \\
k_{33} &= 1 + k(\lambda^2 + n^2)^2
\end{aligned} \tag{4}
$$

Where, $\lambda = \frac{j\pi R}{L}$, $n = \frac{i\pi}{\phi}$, $k = \frac{h^2}{12R^2}$ and h is the thickness of the shell.

The solution of the eigenvalue value problem in equation (3) is the non-dimensional frequency parameter Ω_{ij} and the mode shape coefficients A, B and C for the mode (i,j). The non-dimensional frequency parameter Ω_{ij} can be expressed in terms of the circular natural frequency ω_{ij} as

$$
\Omega_{ij} = \frac{\rho(1-\nu^2)R^2}{E}\omega_{ij}^2 \tag{5}
$$

2.2 Development of Algorithm and Computer Program

A computer program was developed for the determination of natural frequencies and mode shapes through the solution of this eigenvalue problem, for any mode (i,j). The program was developed using the C++ programming language, and the Jacobi method was used for the solution of the symmetric eigenvalue problem due to its simplicity and efficiency [9]. The flowchart for the program is given in Fig. 2.

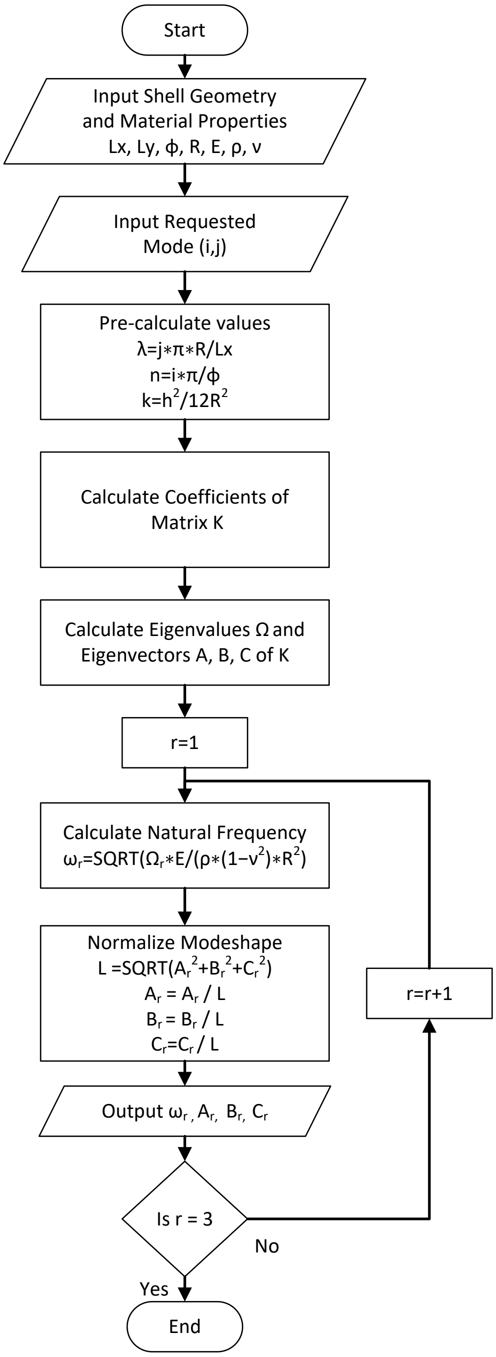

Figure 2: Flowchart for the calculation of natural frequencies and mode shapes.

The solution of the eigenvalue problem for a pair of half-wave numbers (i,j) results in three values of ω_{ij} and three sets of mode shape coefficients, one each for the radial, circumferential and axial modes.

Because the each mode (i,j) can be computed independent of all other modes, this algorithm can easily take advantage of parallel processing for reduced computation time. The solution of each mode (i,j) only requires the computation of the eigenvalues and eigenvectors of a 3×3 matrix so this algorithm is very efficient for computing a large number of modes.

2.3 Validation of the Algorithm

The natural frequencies calculated using the computer program is verified using the results reported by Kuneida et. al. [10], which were also used by Ostovari-Dailamani[8] for verification of the algorithm.The results have been reported for a simply supported panel with R/h=500, $L_y/L_x = 0.5$ and $\phi = \pi/2$. The eigenvalues $\sqrt{\Omega}$ have been reported for 18 circumferential half waves and 9 axial half waves. For the verification, a shell having $L_y = 10m$, $L_x = 20m$, $R = 10/\sqrt{2}m$ was analysed. The non-dimensional frequency parameter $\sqrt{\Omega}$ calculated using the computer program were found to be identical to the reported results.

The results were also verified using the shell geometry studied extensively by Ostovari-Dailamani [8], with $L_x = L_y = 104.8m$, $R = 104.8m$, $R/h = 500$, $\rho hR/E = 10^{-6}s^2$.

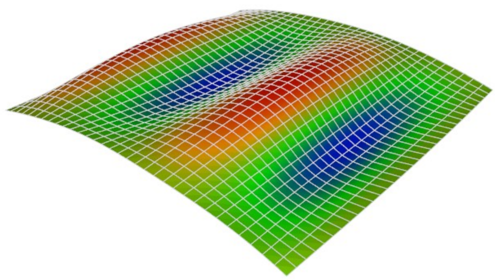

Figure 3: First mode of shell having $L_x = L_y = 104.8m$, $R = 104.8m$, $R/h = 500$, $\rho hR/E = 10^{-6}s^2$. (i,j)=(4,1), f=0.7981 Hz

3. Finite Element Analysis

The Finite Element Method (FEM) is currently the most commonly used method for the static and dynamic analysis of shell structures. The method is based on the sub-division of a field into a finite number of elements connected by nodes in order to approximate the solution of a boundary value problem.

In the finite element method, the natural frequencies and mode shapes are calculated as the solution of the generalized eigenvalue problem

$$[K]\{\phi_i\} = \omega_i^2[M]\{\phi_i\} \tag{6}$$

where [K] is the global stiffness matrix, [M] is the global mass matrix, $\{\phi_i\}$ is an eigenvector and ω_i^2 is the eigenvalue[9]. Many methods are available for the solution of this eigenvalue problem, of which the Lanczos method and the subspace iteration method are the ones most commonly implemented in finite element programs. The Lanczos method is suitable when a large number of modes are required, whereas the subspace iteration method performs much better when only a small number of modes are needed, as in the case of framed structures. Quadrilateral shell elements can be used to discretize singly curved shells, and is recommended for use in the analysis of cylindrical shells [8].

3.1 Shell Element Formulations

Many different types of finite elements have been formulated by researchers, which have been implemented and are available for use in commercial software packages. As it is observed in this study, the choice of element greatly affects the accuracy of the solution.

SAP2000 The shell element used in SAP2000 is a simple element formulated as a combination of a plane stress element to model membrane action and a flat plate element to model bending. According to the CSI Analysis Reference [11], membrane behavior uses an isoparametric formulation that includes translational in-plane stiffness components and a ”drilling” rotational stiffness component in the direction normal to the plane of the element[12].

The four-node quadrilateral element used in this study has 6 degrees of freedom per node, forming a 24×24 stiffness matrix. The CSI Analysis Reference [11] also states that the four nodes need not be coplanar, and the twist is accounted for in the program by considering the angle between the normals at corners.

ABAQUS The element library of ABAQUS has many different shell elements which can be used for meshing [13]. Among the different elements that are available, Ostovari-Dailamani [8] recommends the use of the S8R5 shell element, which is a conventional 8-noded reduced integration shell element based on the Koiter-Sanders shell theory for thick shells, which can be used for static or dynamic analysis. The element has 5 degrees of freedom at each node, with two translational components and three rotation components. The analysis is also carried out with a four noded S4 element for comparison.

ANSYS ANSYS allows selecting the shell element order as either linear or quadratic. The linear option uses the SHELL181 element, which is a four-noded element with 6 degrees of freedom per node, whereas the quadratic option uses the SHELL281 element, which is an eight-noded element, also with 6 degrees of freedom at each node[14].

During the study, the computation of 500 modes using a 40×40 mesh in ANSYS using linear and quadratic elements took 16 seconds and 25 seconds respectively. The computation time is largely dependent on the computer hardware used, but it was observed that a solution using quadratic elements took around 1.5 times longer than by using linear elements.

4. Methodology

The following methodology was followed in this study.

1. The natural frequencies for the roof shell was calculated using the computer program developed. The number of half waves (i,j) was taken between (1,1) and (100,100). The frequencies were then sorted in ascending order.

2. Finite element models were prepared in SAP2000, ANSYS and ABAQUS, with meshes of size 20×20, 40×40 and 80×80 elements. Both linear and quadratic elements are used in ANSYS and ABAQUS.

3. The finite element programs are used to calculate the lowest 500 modes.

4. The percentage error is calculated between the finite element result and the analytical result for each mode.

A reinforced concrete shell having the following parameters is used for this study. The dimensions and material properties were chosen to match with the study by Michiels et. Al. [5].

- Modulus of Elasticity $E = 21.5 \times 10^9 N/m^2$
- Poisson's Ratio $\nu = 0.2$
- Density $\rho = 2400 kg/m^3$
- Length $L_x = L_y = 38m$
- Rise/Span = 0.134
- Radius $R = 38m$
- Thickness $h = 0.08m$
- $\phi = \pi/3$

5. Results

5.1 Analytical Solution

The fundamental natural frequency of the reinforced concrete shell roof was calculated to be 1.4294 Hz, when (i,j)=(4,1). The lowest modes are predominantly out-of-plane modes. Mode 358 is the first in-plane mode with a frequency of 36.069 Hz, which is beyond the 33Hz limit specified in the IS 1893:2016 code for seismic analysis[3].

The distribution of the natural frequencies with the mode number is shown in Fig. 4. It should be noted that the fundamental natural frequency is higher than the frequency content of most earthquakes, which is one of the reasons why roof shells have good performance under earthquake loading.

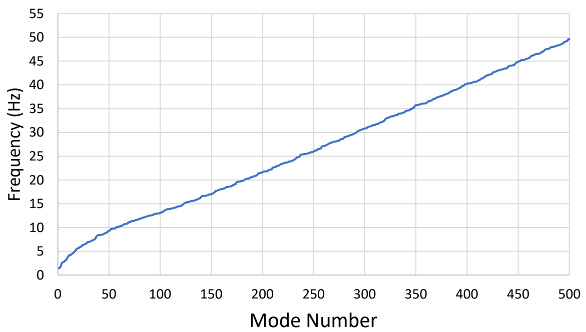

Figure 4: Free vibration frequencies of the cylindrical roof shell calculated using the analytical method.

Table 1: Maximum error in finite element calculated natural frequency compared to the analytical solution.

Mesh	Element	Max. Error %	Mode
20x20	SAP2000	388.82	493
	ANSYS S181	467.87	495
	ANSYS S281	83.54	489
	ABAQUS S4	504.93	495
	ABAQUS S8R5	10.13	491
40x40	SAP2000	10.28	496
	ANSYS S181	43.40	497
	ANSYS S281	3.93	490
	ABAQUS S4	46.06	497
	ABAQUS S8R5	2.97	66
80x80	SAP2000	2.94	318
	ANSYS S181	8.57	491
	ANSYS S281	2.07	256
	ABAQUS S4	8.86	490
	ABAQUS S8R5	2.49	318

5.2 Comparison with Finite Element Analysis

The results from the analytical method were found to closely agree with the finite element results when 8-noded elements and a fine 80×80 mesh was used. The errors for the ABAQUS S8R5 element and the ANSYS SHELL281 element, with an 80×80 mesh was less than 2.5%. This further validates the results obtained from the algorithm developed in this study.

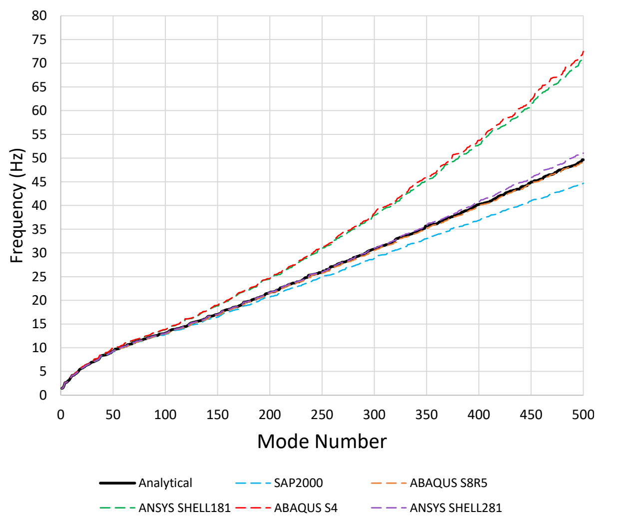

Figure 5: Comparison between the analytical and finite element solution of the natural frequencies using a 40×40 mesh.

5.2.1 Effect of Mesh Size

Large errors were observed in higher modes when using 20×20 mesh divisions.

The number of modes that can be computed with finite element analysis is limited by the number of degrees of freedom, which defines the size of the mass and stiffness matrices. Most of the modes in the first 500 modes studied here are out-of-plane modes, and a coarse mesh can only be used to compute a limited number of them. This causes the finite element programs to compute the in-plane modes instead, which have a much higher natural frequency. This can be clearly seen in the results obtained using a mesh size of 20×20 with SAP2000 as shown in figure 6, where the errors increase rapidly after mode 365. On the other hand, using a higher order element such as the ABAQUS S8R5 can provide acceptable results even with a coarse mesh.

The results were found to be within less than 4% of the analytical soluiton when using quadratic elements with a 40×40 mesh, which is the number of mesh divisions that has been recommended based on mesh convergence studies[5, 8].

5.2.2 Effect of Element Type

The results obtained by using 8-node quadratic shell elements were found to have the least difference to the analytical solution.

The linear elements in both ANSYS and ABAQUS performed poorly in comparison to the 8 node elements. This is most likely due to the curvature of the cylindrical shell, whose effect is not modeled effectively when using 4 node plane quadrilateral elements. The four noded element implemented in SAP2000 considers a small amount of twist through the angle between normals at the nodes, and was found to perform better than the other four node elements, especially with a finer mesh.

The accuracy of linear elements were found to increase as the mesh was refined, and similar levels of accuracy were observed in a 20×20 quadratic mesh and a 80×80 linear mesh. As it can be seen in figure 5, the results obtained using linear elements overestimate the natural frequencies, which would result in a smaller design seismic load when designing using most seismic codes such as IS 1893:2016[3].

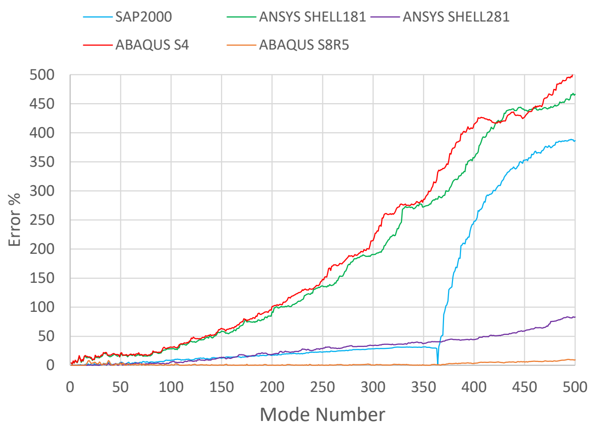

Figure 6: Error in FE computed natural frequencies using 20×20 mesh

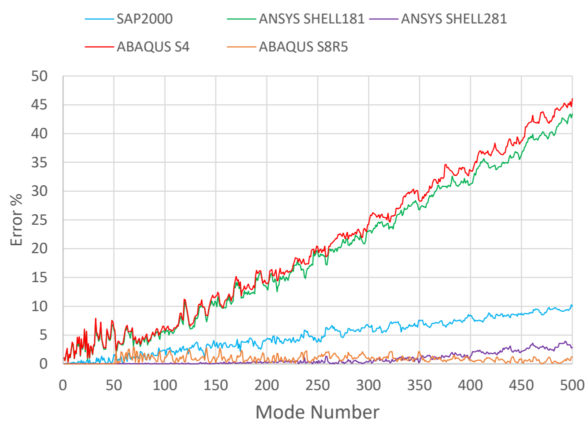

Figure 7: Error in FE computed natural frequencies using 40×40 mesh

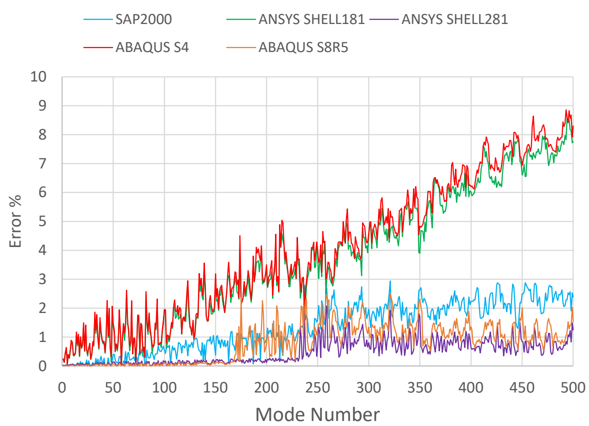

Figure 8: Error in FE computed natural frequencies using 80×80 mesh

6. Conclusions

A highly efficient algorithm was developed for the calculation of natural frequencies and mode shapes of an open circular cylindrical roof shell based on the Goldenveizer-Novozhilov shell theory. The solution calculated using the algorithm is free from errors due to meshing and is only limited by the assumptions of the underlying shell theory. The results were validated using previously reported results.

The results from the analytical method are further validated by the finite element solution obtained by using a fine mesh of quadratic elements. However, it is observed that the natural frequencies computed using the finite element method can be very inaccurate depending on the choice of element and the mesh size. The errors also tend to be higher in higher mode numbers.

Linear shell elements were observed to give inaccurate estimates of the natural frequencies, while providing minimal advantages over quadratic elements in terms of computation time. Therefore, quadratic shell elements should be preferred over linear elements when performing dynamic analysis. The use of linear shell elements which to not take the curvature of the shell into account should be avoided.

It is also recommended to carry out a mesh convergence study to determine the optimum mesh size in order to avoid faulty results when performing dynamic analysis of shells using the finite element method.

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