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Non-linear Steady State Heat Conduction using Element-Free Galerkin Method

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Abstract

Element-free Galerkin (EFG) method is used to solve the non-linear steady state heat conduction for temperature dependent thermal conductivity ($k(T)$). In present work, a one dimensional heat conduction problem with uniform heat generation is solved. Moving least squares (MLS) approximants are used to approximate the unknown function of temperature $T(x)$ with $T^h(x)$ using linear basis and weight functions. Variational method has been used to develop discrete equations. Essential boundary conditions are enforced by using Penalty method. The results has been obtained for one dimensional model using essential MATLAB codes. The results obtained by EFG method are compared with analytical and finite-element method results. The results are also studied by increasing the number of nodes to study the convergence which indicated that EFG has good convergence behaviour. The results are also obtained for different values of scaling parameter (α_s) and any values of α_s between 1.8 and 2.0 was found suitable for providing better results in EFG method.

Keywords

Meshfree, Heat Conduction, Non-linear, EFG, MLS

1. Introduction

Heat conduction is a non-linear phenomenon. Change in material properties with temperature as well as temperature dependent boundary conditions are causes of non-linearity in the conduction. Material properties like thermal conductivity (k), density (ρ) and specific heat (c) are temperature dependent quantities.

The non-linear heat conduction also includes the problems of solid-liquid phase change. Solar energy storage, metal and alloy casting, ice formation and freezing of food stuff are few practical examples where this analysis can be employed.

Conventional mesh-based numerical methods have been widely used in analysis of many physical phenomenon. For the analysis of system involving large deformation, crack propagation, etc. it is necessary to deal with deformation of mesh, which may reduce the accuracy of solution and the processes are also extremely time consuming.

To solve the problems faced with mesh based computational methods meshfree method was developed, which approximates partial differential equations only based on a set of nodes without the need for underlying node connectivity. This method uses a set of nodes scattered within the problem domain as well as sets of nodes scattered on the boundary of domain to represent the problem domain and its boundary.

Meshfree methods was originated at about four decades ago. The first step towards the evolution of meshfree computational methods is Smoothed-Particle Hydrodynamics.[1] On 1992, Nayroles *et al.*[2] used MLS[3] approximation to construct shape function for their new meshfree method, diffused element method (DEM). The research into meshfree methods has become very active after the publication of DEM. Later, Belytschko *et al.*[4] introduced the new method using similar approach as DEM, element-free Galerkin (EFG) method, which also employs the MLS approximation. It uses more accurate numerical integration technique. Due to the superiority of EFG[5], it has been widely used in many problems like modeling of material interfaces [6], fracture mechanics [4],[7] and thin plates and shells[8].

Different mesh based as well as meshfree methods have been used to solve the problems of heat conduction. Donea and Giuliani used FEM based iterative method to solve steady state non linear heat transfer problems with temperature dependent thermal conductivity and radiative heat transfer.[9] Singh used EFG method for composite heat transfer problem where he used different weight function in MLS approximation.[10] Dai *et al.* used local Petrov-Galerkin method to solve transient heat conduction problem which produced the result in good agreement with analytical and finite element methods.[11] Thakur used local Petrov-Galerkin method for phase change problem in his Ph.D. thesis where he used fourth order spline function as weight function in MLS approximation.[12]

This work is an attempt to test EFG method as an alternative to conventional numerical method for non-linear heat conduction problems.

2. Element-free Galerkin Method

Element-free Galerkin method is a meshfree method which only uses set of nodes to construct approximation solution. Unlike mesh based methods like FEM, the connectivity between nodes and shape functions are constructed by the method without recourse to elements.

In EFG method, Galerkin weak form is used to develop discrete system equation. Although the EFG is considered as meshfree with respect to function approximation or construction of shape function, a background mesh is required to perform numerical integration for computing system matrices.

2.1 Meshfree approximation

Creation of shape functions is one of the most important as well as challenging steps in meshfree method, as it is created using scattered nodes without any priori connectivity among them.

A number of ways to construct shape functions have been proposed such as shepard functions[13], smoothed particle hydrodynamics(SPH)[1], moving least square (MLS)[3], radial basis function[14], reproducing kernel particle method[15], partition of unity[16], etc. Among them, MLS is generally considered to be one of the best schemes which is also used in element-free Galerkin method.

2.1.1 Moving Least Square approximation

MLS approximation has two major features:

- Approximated field function is continuous and smooth in the entire problem domain.
- It is capable of producing an approximation with desired order of consistency.

The MLS approximation $T^h(x)$ of the function $T(x)$ is defined in the domain Ω by equation 1[4],

$$T^h(x) = \sum_{i}^{n} p_i(x) a_i(x) = p^T(x)\, a(x) \tag{1}$$

where, n is number of terms in basis, $p_i(x)$ is monomial basis function and $a_i(x)$ is non-constant coefficient.

The coefficient $a_i(x)$ are obtained by minimizing the functional Π[4] given by,

$$\Pi = \sum_{I}^{m} w(x-x_I)[p^T(x_I)a(x) - T_I]^2 \tag{2}$$

where $w(x-x_I)$ is weight function, m is the number of nodes in influence of domain and T_I is nodal parameter at $x = x_I$. This gives

$$A(x)a(x) = B(x)T \tag{3}$$

$$a(x) = A^{-1}(x)B(x)T \tag{4}$$

where A and B are given by,

$$A(x) = p^T(x)w(x-x_I)p(x) \tag{5}$$

$$B(x) = p^T(x)w(x-x_I) \tag{6}$$

Substituting equation 4 into equation 1, we get

$$T^h(x) = \sum_{i}^{n} \Phi_i(x)T_i = \Phi(x)T \tag{7}$$

where $\Phi(x)$ is the MLS shape function and is defined by,

$$\Phi(x) = p^T(x)A^{-1}(x)B(x) \tag{8}$$

2.1.2 Weight function

The weight function $w(x-x_I)$ is non-zero over a small neighbourhood of the node x_I, which is called the domain of influence of node I. Closer nodes have more weighting then the farther nodes. The smoothness of shape function depends on the smoothness of weight function. So, weight function should be selected appropriately. The selected weight function need to satisfy the following conditons:

- $w(x-x_I) > 0$ inside the domain
- $w(x-x_I) = 0$ outside the domain
- $w(x-x_I)$ is monotonically decreasing function

Most commonly used weight functions are *cubic spline, quartic spline, hyperbolic, gussian* weight function etc. In this analysis cubic spline is used as weight function which is given by equation 9.[12]

$$w(x-x_I) = \begin{cases} \frac{2}{3} - 4r^2 + 4r^3 & r \leq \frac{1}{2} \\ \frac{4}{3} - 4r + 4r^2 - \frac{4}{3}r^3 & \frac{1}{2} < r \leq 1 \\ 0 & r > 1 \end{cases} \quad (9)$$

where,

$$r = \frac{|x-x_I|}{dmI}$$

and dmI is the size of the domain of influence of the I^{th} node.

2.2 Enforcement of essential boundary conditions

Like most of the meshfree method, shape function $\Phi(x)$ of EFG method lack Kronecker delta function property, i.e. $\Phi_i(x)$ do not fulfill $\Phi_i(x_j) = \delta_{ij}$. Unlike FEM and FVM, the enforcement of essential boundary condition is difficult in EFG method. Different numerical techniques have been developed to address this problem. Some of the most frequently used techniques are:

- Penalty method
- Lagrange multiplier method
- Direct interpolation method

In this work Penalty method is used to enfroce the essential boundary conditions.

3. Discrete Equation

The governing equation for one dimensional non-linear steady state heat conduction[12] is given by equation 10

$$\frac{\partial}{\partial x}\left[k(T)\frac{\partial T}{\partial x}\right] + Q_g = 0 \qquad in\Omega \quad (10)$$

with boundary conditions,

$$\begin{cases} T = \bar{T} & \text{on}\Gamma_1 \\ k(T)\frac{\partial T}{\partial x} = \bar{q} & \text{on}\Gamma_2 \\ k(T)\frac{\partial T}{\partial x} = h(T_f - T) & \text{on}\Gamma_3 \end{cases} \quad (11)$$

where, T is temperature, Q_g is volumetric heat generation rate, Γ_1, Γ_2 and Γ_3 are boundaries of first, second and third kind, respectively. $\bar{T}$ and $\bar{q}$ are prescribed temperature and heat flux. h is the convective heat transfer coefficient and T_f is environmental temperature.

3.1 EFG formulation

The Galerkin weak formulation of equation 10 is given by

$$\int_\Omega \Phi(x)\left(\frac{\partial}{\partial x}\left[k(T)\frac{\partial T}{\partial x}\right] + Q_g\right) d\Omega = 0 \quad (12)$$

Integrating above equation by parts gives

$$\int_\Omega \frac{\partial \Phi(x)}{\partial x} k(T)\frac{\partial T}{\partial x} d\Omega - \int_\Gamma \Phi(x)k(T)\frac{\partial T}{\partial x} d\Gamma = \int_\Omega Q_g \Phi(x) d\Omega \quad (13)$$

Using Penalty method to enforce essential boundary condition equation (13) can be written as

$$[\bar{K}(T)]\{T\} = \{f\} \quad (14)$$

where,

$$\bar{K} = \int_\Omega \frac{\partial \Phi(x)}{\partial x} k(T)\frac{\partial T}{\partial x} d\Omega + \int_{\Gamma_3} h\Phi(x)T d\Gamma + \int_{\Gamma_1} \Phi(x)\alpha\Phi(x) d\Gamma \quad (15)$$

$$f = \int_\Omega Q_g \Phi(x) d\Omega + \int_{\Gamma_3} h\Phi(x)T d\Gamma + \int_{\Gamma_1} \Phi(x)\alpha\bar{T} d\Gamma \quad (16)$$

α is penalty factor which is generally a large value $(10^4 - 10^{13})$.

3.2 Solution of non-linear system

For such nonlinear problems, an iterative procedure is required. A *predictor-corrector*[12] scheme based on direct substitution iteration has been applied in the current analysis which has the following form:

Predictor step

$$K(T_0)T_* = f \tag{17}$$

Corrector step

$$K(T_{\bar{p}})T_{p+1} = f \tag{18}$$

where T_0 is initial guess; $T_{\bar{p}} = \gamma T_p + (1-\gamma)T_{p-1}$, $\gamma \in (0,1)$; $T_1 = T_*$ and $p = 1,2,\ldots$ iteration counter.

4. Methodology

The analysis of non-linear heat flow in a rod with temperature dependent thermal conductivity is done. There is no loss of heat from its surfaces and ends of rod are maintained at equal temperature ($T_a = T_b = 0°C$). Heat is produced at a volumetric rate of Q_g.

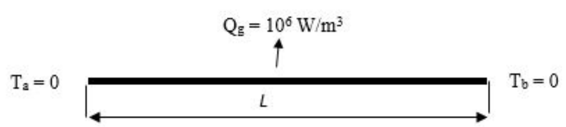

Figure 1: Heat conduction through uniform bar

Relevent parameters used in analysis is listed in Table 1

Table 1: Data for non-linear heat conduction[12]

Parameter	Values
Length(L)	1m
Thermal conductivity(k)	$k(T) = k_0(1-0.0005T)$
	$k_0 = 400W/m°C$
Uniform heat (Q_g)	$10^6 W/m^3$

The analytical solution of the problem in steady state condition is given by[17]

$$T(x) = \frac{-1+\sqrt{1-0.0005\frac{Q_g}{k_0}(Lx-x^2)}}{-0.0005} \tag{19}$$

Figure 2 shows the temperature profile along the bar computed using EFG along with its analytical result. The result obtained from the EFG method is compared against the results from analytical and FEM solution at some typical location in Table 2 . This result shows that EFG method competes well with FEM and has result in good agreement with analytical solution.

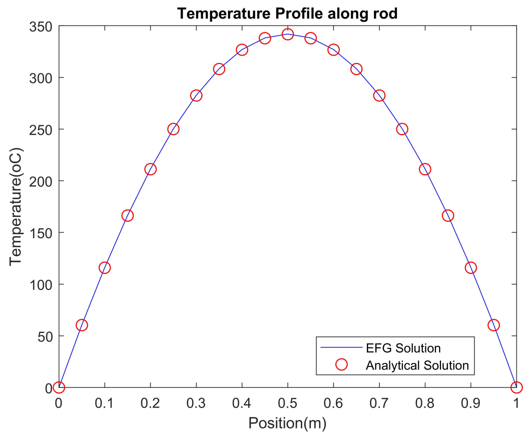

Figure 2: Temperature variation along the length of bar for non-linear steady state heat conduction

Table 2: Comparison of EFG solution with FEM and analytical solution for one dimensional non-linear steady state heat conduction problem

Position x(m)	Temperature (°C)		
	Analytical	FEM	EFG
0.2	211.14	211.13	211.16
0.3	282.44	282.40	252.46
0.4	326.67	326.60	326.70
0.5	341.68	341.59	341.70

This analysis was carried out for different size of the support domain. The size of the support domain[5] is given by the relation

$$r_s = \alpha_s d_n \tag{20}$$

where r_s is radius of support domain, α_s is scaling parameter and d_n is characteristic length between nodes.

The values of relative error based on L_2 norm is plotted in Figure 3 with respect to the scaling parameter of the corresponding domain. The values of relative error are less for any value of α_s between 1.8 and 2.0 which can be used to obtain more accurate result than conventional mesh based method. Larger size of the support domain may not maintain the local characteristics of the approximation which could deviate the resulting solutions from its true value as seen in the plot, especially for the value of α_s greater than 2.0.

Relative error is also calculated with increasing number of nodes. It is observed that with increase in number of nodes in the domain the value of relative error decreases (Figure 4). This indicates that EFG has good convergence behaviour.

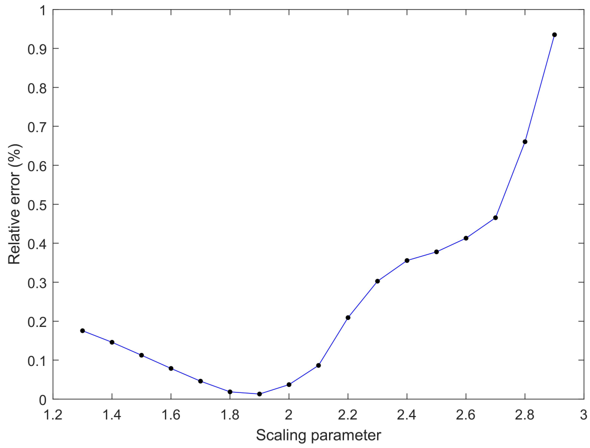

Figure 3: Variation of relative error with size of local domain

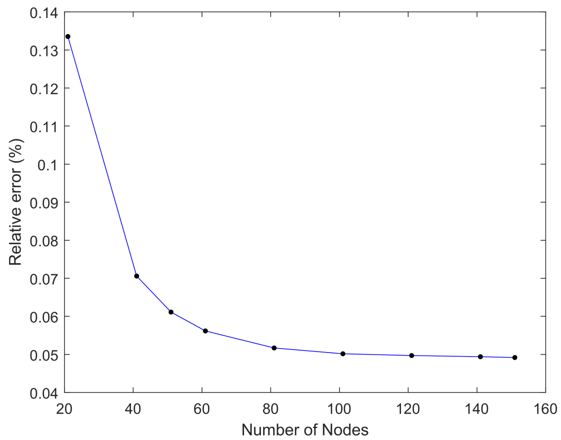

Figure 4: Relative error with number of nodes for one dimensional non-linear steady state problem

5. Conclusion

The EFG method can be used to solve non-linear steady state heat conduction problem using penalty method to enforce essential boundary conditions. Selection of size of support domain palys important roles while solving any problem using meshfree method. It is observed from error analysis that the value of scaling parameter (α_s) between 1.8 and 2.0 provides more favorable solutions. With proper selection of support domain EFG can provide numerical solution of nonlinear heat conduction problem with higher accuracy than other conventional mesh based methods. This method can be used as alternative to FEM to solve different nonlinear problems.

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