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Free Vibration Analysis of Selected Pelton Turbine using Dynamic Approach

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Abstract
This research work was focused on the modeling of the Pelton turbine unit and has covered dynamic behavior of the centrally located rigid runner on the circular flexible shaft, which was supported by the rigid bearings on both ends and enabled to determine the natural frequency of the system by using different models and compared with the continuous system model. The unit was modeled as a discrete and continuous system. Föppl/Jeffcot rotor models and Rayleigh's energy method: Effective mass models were used for the discrete system models. The model for the continuous system was developed by calculating the kinetic and potential energy of the runner-buckets assembly and shaft. The governing equations were formulated by using Lagrange's equation and solved analytically for natural frequency by using Rayleigh-Ritz method. The critical frequencies were determined for developed models based on the real Pelton turbine unit with the capacity of 2 kW at the rated 1500 rpm. The discrete system model by considering the effective mass of shaft which was simply supported at ends determined the critical frequency of the unit without significant deviation for the continuous shaft-runner-buckets system model.

Keywords
Rotor – discrete – continuous – model – natural frequency

1. Introduction

Rotor dynamics study the lateral/transverse (bending), longitudinal (axial), and torsional vibration of the rotating shafts with the objective of limiting the vibration under an acceptable range and minimizing the probability of failure due to vibration. The possible forces responsible for the vibration increase in a hydro turbine may be mechanical, hydraulic or electrical[1]. Dynamic behavior of the rotors can be studied and predicted to some extent by appropriate analysis of their dynamic response through proper mathematical modeling of physical system. Study of dynamic response of the Pelton turbine unit, one of the widely used water turbines worldwide, may contribute for the improvement in the performance as well as reliability, stability and the life span of the components of the hydraulic power system. If the frequency of external excitation coincides with one of the natural frequencies of the system, resonance occurs leading to the dangerously large oscillations and may cause excessive deflection and failure. Sometimes the failure of structures like buildings, bridges, turbines, and airplane wings etc. are associated with the occurrence of resonance[2]. Thus, the calculation of the natural frequencies of any system is one of the important part of vibration study and analysis.

The model for the study of dynamic behavior of rotor and rotating parts was first developed by the German engineer August Föppl in 1895 and American Henry Homan Jeffcott in 1919. This model is commonly known as the Föppl/Jeffcot rotor, or simply Jeffcot rotor, which consisted of a single rigid disk, centrally located on a flexible shaft of constant circular cross-section supported by bearings placed at each end of the shaft[3]. A mathematical model was developed by using Föppl-Jeffcott rotor model for finding the natural frequency of system for damped and undamped free vibration and the general solutions for both cases were also formulated[4].

The study of dynamic behavior of Pelton turbine assembly was carried out to obtain the natural frequency and response of mass unbalance of runner on amplitudes of vibration by analytically and numerical

simulation with finite element method code ANSYS. The critical frequencies evaluated by analytical and numerical methods were compared[5]. They modeled and dynamically calculated the natural frequency and mode shapes of 23 MW Pelton runner by FEM code ANSYS. The mode shapes of the unit were investigated experimentally by impulse test in the field and the natural frequencies were calculated by the online FFT PC program[6].
The main shaft system in the hydro-turbine generating unit was modeled by using ANSYS finite element software. Their research work concluded that as the operating speed of the shaft increases, the natural frequency of the positive whirl increases and negative whirl decrease [7]. Detuning procedure was recommended to prevent resonances of Pelton runner water turbines. By detuning of the buckets, the natural frequency of the system was increased without much affecting the stiffness of the buckets and the structural integrity of the runner[8].
Most of the research works related to the turbine blade vibration has been carried out on gas and steam turbines. Studies related to effect of sediments and erosion of buckets of the Pelton turbine has been largely conducted by many researchers, but limited work has been done regarding the dynamic behavior of Pelton turbines. For small scale, i.e. mini and micro hydro power plants, turbines are generally designed and manufactured locally in most of the countries without proper research and tests, which may result to higher level of vibrations. Thus, the dynamic analysis of the locally manufactured turbines is necessary for reliable performance and stability. This research work was focused on the dynamic behavior of the Pelton turbine unit of capacity 2 kW at the rated 1500 rpm and investigated the natural frequencies of the unit by different models and compared with the continuous system model.

2. Mathematical Model Development and Analytical Solution

Solution of most engineering problems often require mathematical modeling of a physical system. The mathematical models are the governing equations for good understanding of the different characteristics of the real-world physical systems. The basic components considered for the model development were rigid runner-buckets assembly (disk) and the flexible shaft supported by rigid bearings at both ends. The models were developed considering the system as discrete and continuous systems and focused for the output of natural frequencies.

Assumptions

The rotor-bearing system was operating at steady state condition and the damping in the rigid and isotropic bearings and seals were neglected. The mass unbalanced on the runner-buckets assembly was negligible and the runner-buckets assembly was treated as rigid disk. The displacement in the shaft axial direction was neglected.

2.1 Discrete System Models

Föppl/Jeffcot rotor model and Rayleigh's energy method: Effective mass model were used for the discrete system models.

2.1.1 Föppl/Jeffcot Rotor Models

Considering the single rigid disk mounted on axial center of a circular flexible shaft, which was supported by rigid bearings at each end. The mass of the shaft was assumed to be negligible compared to that of the disk and the geometric center of the disk coincided with the shaft center line. With the assumption that the rotor-disk didn't not affect the stiffness of the mass-less shaft, the lateral bending stiffness at the axial center of the simply supported uniform beam was given by,

$$K_s = \frac{48EI}{L^3} \tag{1}$$

Where, K_s - stiffness of the shaft, E - the elastic modulus of the shaft, L - the length of shaft between the bearings, I - area moment of inertia. If D - the diameter of the uniform cylindrical shaft, then, $I = \frac{\pi D^4}{64}$

The undamped natural frequency of the system for the simply supported shaft at ends was given by the expression as [4],

$$\omega_n = \sqrt{\frac{K_s}{M_D}} = \sqrt{\frac{48EI}{M_D L^3}} \tag{2}$$

Where, M_D - the mass of the disk (runner-buckets).
Similarly, the undamped natural frequency for the shaft with fixed supports at both ends was represented as [2],

$$K_s = \frac{192EI}{L^3} \tag{3}$$

$$\omega_n = \sqrt{\frac{K_s}{M_D}} = \sqrt{\frac{192EI}{M_D L^3}} \tag{4}$$

2.1.2 Rayleigh's Energy Method: Effective Mass Models

A reasonable equivalent one degree of freedom system was developed by approximating the inertia effects of the elastic member shaft of the unit, i.e. effective mass of shaft. The principle of conservation of energy (kinetic and strain energy) was used to evaluate the effective mass of the shaft. Shaft with simple supports at ends and equivalent spring mass configuration were formulated in this model. For the simply supported uniform shaft, considering the effective mass of the shaft, the undamped natural frequency of the system was expressed as [9],

$$\omega_n = \sqrt{\frac{K_s}{(M_D + 0.4857 m_s)}} \tag{5}$$

Where, m_s - the mass of the shaft.
The natural frequency of the spring-mass system, considering the effective mass of spring and assuming the shaft to be rotating in simple harmonic motion, was defined as [10],

$$\omega_n = \sqrt{\frac{K_s}{\left(M_D + \frac{m_s}{3}\right)}} = \sqrt{\frac{48EI}{\left(M_D + \frac{m_s}{3}\right) L^3}} \tag{6}$$

2.2 Continuous System Models

The mathematical model was developed by calculating the kinetic and strain energy of the assembly. The equations of motion derived by using Lagrange's equation were solved for natural frequencies by Rayleigh-Ritz analytical solution method.
The model was developed considering flexible continuous and distributed mass shaft and lumped mass of disk concentrated at the middle point of the shaft which was simply supported at ends. Euler's angles were used to describe the configuration of the shaft. The motion of the unit was defined by using Euler's angles, which gave the angular velocity at any instant of time in the form of Euler's angles. The kinetic energy of disk and kinetic and strain energy of shaft were represented in the form of these angles. Sum of kinetic and strain energy was the total energy of the system. Then, finally these energies in the form of Euler's angles were converted to independent generalized coordinates by applying Lagrange's equation.

2.2.1 Rotational Matrix for 3-1-2 Euler Angles

Any rotation can be described by three successive rotations about linearly independent axes and these rotations are Euler angles. X, Y and Z were fixed inertia frame coordinates and x, y and z were the rotor (body) fixed coordinates. The sequence of the rotations of the axes were z, x and y (3-1-2). Firstly, the rotation of body was about Z-axis by angle ϕ, and by angle θ about x_1 intermediate axis and then by angle ψ about y_2 (y) body fixed axis [11]. Finally, the resultant rotational matrix for 3-1-2 Euler angles was expressed as,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = R \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \cos\phi\cos\psi - \sin\phi\sin\theta\sin\psi & \sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi & -\cos\theta\sin\psi \\ -\sin\phi\cos\theta & cos\phi\cos\theta & \sin\theta \\ \cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi & \sin\phi\sin\psi - \cos\phi\sin\theta\cos\psi & \cos\theta\cos\psi \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{7}$$

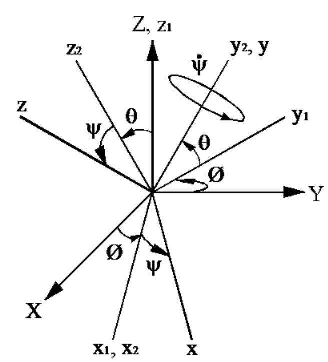

Figure 1: Rotation about axes

Where, R was the rotational matrix.

2.2.2 Angular Velocity of xyz Frame

The instantaneous angular velocity of the xyz frame was,

$$\omega = \dot{\phi}Z + \dot{\theta}x_1 + \dot{\psi}y \tag{8}$$

Where, Z, x_1, and y were unit vectors along the axes Z, x_1, y respectively.
The components of the angular velocities were found to be,

$$\begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix} = \begin{bmatrix}-\dot{\phi}\cos\theta\sin\psi + \dot{\theta}\cos\psi\\ \dot{\phi}\sin\theta + \dot{\psi}\\ \dot{\phi}\cos\theta\cos\psi + \dot{\theta}\sin\psi\end{bmatrix} \tag{9}$$

For the system considered, the spinning axis was Y and angular motion about X and Z axes were comparatively small [12]. Thus $\cos\theta \approx 1, \cos\phi \approx 1, \sin\theta \approx \theta, \sin\phi \approx \phi$ and then as per Equation 9,

$$\begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix} \cong \begin{bmatrix}-\dot{\phi}\sin\psi + \dot{\theta}\cos\psi\\ \dot{\phi}\theta + \dot{\psi}\\ \dot{\phi}\cos\psi + \dot{\theta}\sin\psi\end{bmatrix} \tag{10}$$

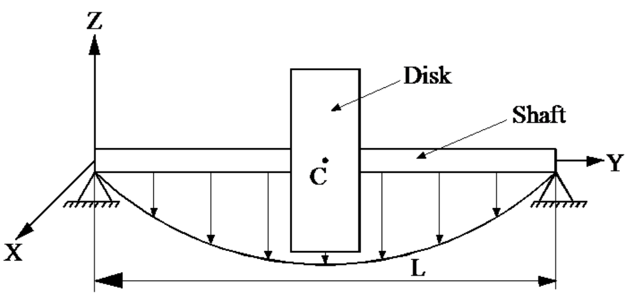

Figure 2: Shaft and Runner-buckets Assembly

2.2.3 Disk

As the disk was the rigid component of the vibratory system, it possesses kinetic energy only. There was no translation motion of the disk center along Y-axis of the inertial frame. However, the displacement of the disk's center of mass along the direction of X and Z axes were designated by u and w respectively.
Thus, the kinetic energy of the disk was given by[13],

$$T_D = \frac{1}{2}M_D\left(\dot{u}^2 + \dot{w}^2\right) + \frac{1}{2}\left[I_{Dxx}\omega_x^2 + I_{Dyy}\omega_y^2 + I_{Dzz}\omega_z^2\right] \tag{11}$$

Where, T_D - the kinetic energy of disk.
M_D - the mass of the disk.
I_{Dxx}, I_{Dyy} and I_{Dzz} were the moment of inertia about the principal axes, X, Y and Z respectively. As the disk was thin (as thickness L_D/R_D radius, was very small) and assumed to be symmetrical, then $I_{Dxx} = I_{Dzz} = M_D R_D^2/4$ and $I_{Dyy} = M_D R_D^2/2$.
Then, substituting the values of components of angular velocities in Equation 11.

$$T_D = \frac{1}{2}M_D\left(\dot{u}^2 + \dot{w}^2\right) + \frac{1}{2}I_{Dxx}\left[\left(\dot{\phi}^2\sin^2\psi + \dot{\theta}^2\cos^2\psi\right)\right.$$
$$\left.\left(\dot{\phi}^2\cos^2\psi + \dot{\theta}^2\sin^2\psi\right)\right] + \frac{1}{2}I_{Dyy}\left(\dot{\phi}^2\theta^2 + \dot{\psi}^2 + 2\dot{\phi}\theta\psi\right) \tag{12}$$

As angles θ and ϕ were very small and the shaft was assumed to be running with the constant angular velocity Ω, which implies that acceleration in the circumferential direction was not considered, i.e. $\dot{\phi}^2\theta^2 \approx 0, \dot{\psi} = \Omega$ and Equation 12 resulted as,

$$T_D = \frac{1}{2}M_D\left(\dot{u}^2 + \dot{w}^2\right) + \frac{1}{2}I_{Dxx}\left(\dot{\theta}^2 + \dot{\phi}^2\right) + \frac{1}{2}I_{Dyy}\left(\Omega^2 + 2\dot{\phi}\theta\Omega\right) \tag{13}$$

2.2.4 Shaft

The shaft was considered as a flexible element with constant circular cross-section and it possesses both kinetic and strain energies. The K.E. of the shaft was defined for an element and integrated over the length of the shaft 'L'. The K.E. of the shaft was given by the expression and substituting the components of angular velocity resulted as [13],

$$T_S = \frac{\rho A}{2}\int_0^L\left(\dot{u}^2 + \dot{w}^2\right)dy + \frac{\rho I_{xx}}{2}\int_0^L\omega_x^2 dy + \frac{\rho I_{yy}}{2}\int_0^L\omega_y^2 dy + \frac{\rho I_{zz}}{2}\int_0^L\omega_z^2 dy$$
$$T_s = \frac{\rho A}{2}\int_0^L\left(\dot{u}^2 + \dot{w}^2\right)dy + \frac{\rho I}{2}\int_0^L\left(\dot{\theta}^2 + \dot{\phi}^2\right)dy + \rho I L\Omega^2 + 2\rho I\Omega\int_0^L\theta\dot{\phi}\,dy \tag{14}$$

Where,

T_s-kinetic energy of the shaft,

$\rho-$ the mass per unit volume,

A$-$the cross- sectional area of shaft and it was assumed to be constant

$I-$the area moment of inertia of the shaft cross-section about the neutral axis and it was also supposed to be constant (i.e., for symmetrical shaft, $I_{xx} = I_{zz} = I = \pi D^4/64, D$ - diameter of the shaft).

Strain energy is the form of potential energy associated with the elasticity of a deformable body. Potential energy of the shaft is given by the relation of strain energy. The expression for the strain energy of the shaft was expressed as [14],

$$U_s = \frac{EI}{2}\int_0^L \left[\left(\frac{\partial^2 u}{\partial y^2}\right) + \left(\frac{\partial^2 w}{\partial y^2}\right)^2\right] dy \tag{15}$$

Where,

U_s-Strain energy of the shaft.

$E-$the Young's modulus of elasticity of the shaft material.

$I-$for the symmetrical shaft, $I_{xx} = I_{zz} = I$

Total kinetic energy of the system was,
$T = T_D + T_s$
Substituting the values from Equation 13 and 14, total kinetic energy was resulted as,

$$\begin{aligned} T = {} & \frac{1}{2}M_D\left(\dot{u}^2+\dot{w}^2\right) + \frac{1}{2}I_{Dxx}\left(\dot{\theta}^2+\dot{\phi}^2\right) + \frac{1}{2}I_{Dyy} \\ & \left(\Omega^2 + 2\dot{\phi}\theta\Omega\right) + \frac{\rho A}{2}\int_0^L \left(\dot{u}^2+\dot{w}^2\right) dy + \frac{\rho I}{2}\int_0^L \\ & \left(\dot{\theta}^2+\dot{\phi}^2\right) dy + \rho I L \Omega^2 + 2\rho I \Omega \int_0^L \theta\dot{\phi}\, dy \end{aligned} \tag{16}$$

Total potential energy of the system was equal to the strain energy of shaft only, according to Equation 15, it was represented as[14],

$$U = \frac{EI}{2}\int_0^L \left[\left(\frac{\partial^2 u}{\partial y^2}\right) + \left(\frac{\partial^2 w}{\partial y^2}\right)^2\right] dy \tag{17}$$

The bearings at the support were considered as rigid and isotropic with negligible damping.

2.2.5 Lagrange's Equation

Using Lagrange's equation for the system of rigid bodies in the form [14];

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Fq_i \tag{18}$$

Where,
n - the number of degrees of freedom $(1 \le i \le n)$
q_i - the system's generalized independent coordinates.
Fq_i - the generalized forces.

2.2.6 Rayleigh-Ritz Method of Analytical Solution

The shape function of the system was equal to the one of the first mode of vibration of a shaft [12]; i.e.

$$f(y) = \sin\left(\frac{\pi y}{L}\right) \tag{19}$$

The expressions for the displacements u and w in X and Z directions respectively were expressed as,

$$u(y,t) = f(y)\, q_1(t) = q_1 \sin\left(\frac{\pi y}{L}\right)$$

$$w(y,t) = f(y)\, q_2(t) = q_2 \sin\left(\frac{\pi y}{L}\right)$$

Where, q_1 and q_2 were generalized independent coordinates. As the angular displacements ϕ and θ were small, they were approximated as,

$$\theta = \frac{\partial w}{\partial y} = \frac{df(y)}{dy}q_2 = g(y)\, q_2 = \frac{\pi}{L}\cos\left(\frac{\pi y}{L}\right) q_2$$

$$\phi = -\frac{\partial u}{\partial y} = -\frac{df(y)}{dy}q_1 = -g(y)\, q_1 = -\frac{\pi}{L}\cos\left(\frac{\pi y}{L}\right) q_1$$

2.2.7 Total kinetic Energy of Shaft-Disk Assembly

Therefore, the kinetic energy of the shaft-disk assembly as per Equation 16 became,

$$\begin{aligned} T = {} & \frac{1}{2}\left[M_D + \frac{\rho AL}{2} + \frac{\rho I \pi^2}{2L}\right]\left(\dot{q}_1^2+\dot{q}_2^2\right) - \frac{\rho I \Omega \pi^2}{L}\dot{q}_1 q_2 \\ T = {} & \frac{1}{2}m\left(\dot{q}_1^2+\dot{q}_2^2\right) - \alpha\Omega\dot{q}_1 q_2 \end{aligned} \tag{20}$$

Where,

$$m = M_D + \frac{\rho AL}{2} + \frac{\rho I\pi^2}{2L} \tag{21}$$

$$\alpha = \frac{\rho I\pi^2}{L} \tag{22}$$

2.2.8 Total Strain Energy of Shaft-Disk Assembly

The strain energy of the shaft and the overall system according to Equation 17 resulted to,

$$U = \frac{1}{2}K\left(q_1^2 + q_2^2\right) \tag{23}$$

Where,

$$K = \frac{\pi^4 EI}{2L^3} \tag{24}$$

Using Lagrange's equation for the obtained total kinetic energy in Equation 20 and strain energy in Equation 23 of the system in the form of q_1 and q_2 generalized coordinates for free vibration with the balanced disk, the equation of motion was represented as,

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \Omega\begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{25}$$

2.3 Analytical Solution

The Newton's equations of motion according to Equation 25 was represented as,

$$\begin{aligned} m\ddot{q}_1 - \alpha\Omega\dot{q}_2 + Kq_1 = 0 \\ m\ddot{q}_2 - \alpha\Omega\dot{q}_1 + Kq_2 = 0 \end{aligned} \tag{26}$$

The solution to the above systems of homogeneous second order differential equations take the form as,

$$q_1 = Q_1 e^{\lambda t}$$

$$q_2 = Q_2 e^{\lambda t}$$

Where, Q_1 and Q_2 denoted the mode shape (also called the Eigen function or normal mode) and λ was the scalar. Here, λ describes the time variation and $Q_{1,2}$ denotes the amplitudes of the vibration in the form of generalized coordinate's $q_{1,2}$.
By substituting the values of Q_1 and Q_2 in Equation 26, for non-trivial solution (the determinant of the coefficient matrix should have the value equal to zero) and solving for λ resulted as,

$$\left(m\lambda^2 + K\right)^2 + (\alpha\Omega\lambda)^2 = 0$$

$$m^2\lambda^4 + \left(2mK + \alpha^2\Omega^2\right)\lambda^2 + K^2 = 0 \tag{27}$$

The solution of the above equation for the complex constant λ, gave;

$$\lambda_n = \pm j\omega_n \tag{28}$$

For a single degree of freedom system, it has one natural frequency, and for n degrees of freedom system, there are n natural frequencies. The general relationship between ω and Ω is,

$$\omega = s\Omega \tag{29}$$

Then, according to Equation 28,

$$\lambda = \pm j\omega = \pm s\Omega j$$

Substituting the value in Equation 27 and solving for Ω, the results were obtained as,

$$\Omega_1 = \sqrt{\frac{K}{s\left(sm + \alpha\right)}} \tag{30}$$

$$\Omega_2 = \sqrt{\frac{K}{s\left(sm - \alpha\right)}} \tag{31}$$

The undamped critical speed of the system is defined as,

$$\Omega_{cr} = \omega_n \tag{32}$$

Where,

Ω_{cr} – Critical speed of the shaft rotation and

ω_n – Undamped natural frequency of the system.

Then, from Equation 29 and 32, for natural frequency or critical speed of the system, s = 1,
Substituting the value of s in Equation 30 and 31 which resulted as,

$$\begin{aligned} \Omega_{1cr} = \omega_{1n} = \sqrt{\tfrac{K}{(m+\alpha)}} \\ \Omega_{2cr} = \omega_{2n} = \sqrt{\tfrac{K}{(m-\alpha)}} \end{aligned} \tag{33}$$

Where,
Ω_{cr1} – Critical speed of the shaft for backward whirl
Ω_{cr2} – Critical speed of the shaft for forward whirl

3. Results and Analysis

The developed mathematical models were solved analytically to find the natural frequencies under undamped free vibration condition of the system. The results obtained from the analytical solutions by using the different developed mathematical models were then analyzed. The developed mathematical models were solved for the undamped natural frequencies of the Pelton turbine unit installed with following specifications.

Table 1: Parameters used for the calculation of critical frequency

Parameters	Values
Output Power	2000 W
Rated RPM	1500 rpm
Pitch circle diameter, PCD	155 mm
Number of Buckets	16
Total Mass of Runner- Buckets Assembly, M_D	10.654 kg
Mass of shaft, m_s	3.281 kg
Diameter of Shaft, D	32 mm
Length of Shaft(bearing to bearing center), L	519 mm
Density of shaft Material, ρ	7860 kg/m^3
Young's Modulus of Elasticity of the Shaft Material, E	202 GPa

According to Equation 21 and 22, the values of m and α were calculated to be,

$$m = 12.298kg$$

$$\alpha = 7.694 \times 10^{-3} kg$$

3.1 Natural Frequency from Föppl/Jeffcot Rotor Model

According to the Equation 2, the undamped natural frequency of the unit when both ends of the shaft were simply supported was calculated as,
$\omega_n = 578.86 \quad \text{rad}/\sec$
Similarly, for shaft with fixed supports at ends,
$\omega_n = 1157.721 \quad \text{rad}/\sec$

3.2 Natural Frequency From Rayleigh's Energy Method

For the simply supported uniform shaft, also considering the effective mass of the shaft, the undamped natural frequency of the system was found out by Equation 5 as,
$\omega_n = 539.89 \quad \text{rad}/\sec$
The natural frequency of the equivalent spring-mass system, considering the effective mass of spring and assuming the shaft to be rotating in simple harmonic motion was determined as per the Equation 6 to be,
$\omega_n = 551.257 \quad \text{rad}/\sec$

3.3 Natural Frequencies from Continuous System Model

For critical frequency, under the rotating condition Ω (rotating speed of the shaft and disk) and the corresponding angular frequencies were equal, i.e. $\omega_n = \Omega_{cr}$. As per the Equation 24 , the value of K was evaluated to be,
$K = 3622335 \quad N/m$
Then, according to Equation 33, the values for critical frequencies were calculated as,
$\Omega_{1cr} = \omega_{1n} = 542.552 \quad \text{rad}/\sec$
Also,
$\Omega_{2cr} = \omega_{2n} = 542.891 \quad \text{rad}/\sec$
For the operating speed of N = 1500 rpm, the angular frequency of the Pelton turbine unit came to be,
$\omega_{ops} = \frac{2\pi N}{60} = 157.08 \quad \text{rad}/\sec$
The analytical results from different models of Pelton turbine unit installed showed that the critical frequency of the unit determined was well above the operating range and in the very safe region. The critical speed of the unit from continuous system modeling was 5181 rpm for backward whirl, which was very high to the operating speed of the turbine, i.e. 1500 rpm. These results were may be due to the large diameter of the shaft, as the design of the Pelton unit was done taking high value of factor of safety.
The natural frequency of the unit by continuous system modeling was 542.552 rad/sec and 542.891 rad/sec for backward and forward whirl. Its values by using Fö ppl/Jeffcot rotor models were 578.86 rad/sec and 1157.721 rad/sec for simple supports and fixed supports at ends of the shaft respectively. Similarly, the value of critical frequencies by Rayleigh's energy method:

Effective mass models was calculated to be 539.89 rad/sec and 551.257 rad/sec for shaft which was simply supported at ends and equivalent spring-mass configuration respectively.
Rajak et al. (2014) evaluated the natural frequency of the 3 kW Pelton turbine unit at rated 1500 rpm to be 137.86 and 137.98 Hz for backward and forward whirl by numerical simulation with finite element method code ANSYS. By using the developed mathematical model for continuous system, the value of natural frequency for the given turbine has been determined to be 136.144 and 136.34 rad/sec for backward and forward whirl respectively. It has been investigated that the error was not significant, i.e. 1.245 % and 1.189% for backward and forward whirl respectively.

4. Conclusions

The mathematical models for the real Pelton turbine unit were developed in the form of discrete system models and continuous system model. For discrete system models Föppl/Jeffcot rotor model and Rayleigh's energy method: Effective mass model were used. By calculating the kinetic and strain energy of the shaft and disk, the governing equations of motion were developed for continuous system models. The equations of motion derived by using Lagrange's equation were solved for natural frequencies by Rayleigh-Ritz analytical solution method. The natural frequency calculated by modeling the Pelton turbine unit as single degrees of freedom discrete system by considering the effective mass of simply supported shaft at ends was 539.89 rad/sec, which was close to the natural frequency by multi-degrees of freedom continuous system model, i.e. for backward and forward whirl 542.552 rad/sec and 542.891 rad/sec respectively. Approximated discrete system for continuous shaft and disk (runner-buckets) system provided the output natural frequency of the unit with no significant deviation.

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References

[1] Yulin Wu, Shengcai Li, Shuhong Liu, Hua-Shu Dou, and Zhongdong Qian. *Vibration of hydraulic machinery*. Springer, 2013.

[2] Singiresu S Rao and Fook Fah Yap. *Mechanical vibrations*, volume 4. Prentice Hall Upper Saddle River, 2011.

[3] FC Nelson. Rotor dynamics without equations. *International Journal of COMADEM*, 10(3):2, 2007.

[4] Robert Surovec, Jozef Bocko, and Juraj Šarloši. Lateral rotor vibration analysis model. *American Journal of Mechanical Engineering*, 2(7):282–285, 2014.

[5] Aman Rajak, Prateek Shrestha, Manoj Rijal, Bishal Pudasaini, and Mahesh Chandra Luintel. Dynamic analysis of pelton turbine and assembly. In *Proceedings of IOE Graduate Conference 2014, Nepal,*.

[6] Emanuel Pesatori, Giorgio Turozzi, Paolo Pennacchi, and Alessandro Tosi. Dynamic investigation on a pelton runner: Fem calculation and experimental results. In *Proceeding of the ASME 2007 International Design Engineering Technical Conferences & Computer and Information in Engineering Conference, Las Vegas, Nevada, USA*, 2007.

[7] Bing Bai, Lixiang Zhang, Tao Guo, and Chaoqun Liu. Analysis of dynamic characteristics of the main shaft system in a hydro-turbine based on ansys. *Procedia Engineering*, 31:654–658, 2012.

[8] Joachim Schmied, Thomas Weiss, and Richard Angehrn. Detuning of pelton runners. In *7th IFToMM Conference on Rotor Dynamics*, 2006.

[9] *Theory of Vibration with Applications*. Pearson Education Asia Limited, 2005.

[10] *Mechanical Vibrations: Theory and Applications*. Cengage Learning, USA, 2012.

[11] H Alemi Ardakani and TJ Bridges. Review of the 3-2-1 euler angles: a yaw-pitch-roll sequence. *Department of Mathematics, University of Surrey, Guildford GU2 7XH UK, Tech. Rep*, 2010.

[12] Pedro Paulo. A time-domain methodology for rotor dynamics: Analysis and force identification. *Univeridade Tecnica de Lisboa, Portugal*, 2011.

[13] SAA Hosseini and SE Khadem. Free vibrations analysis of a rotating shaft with nonlinearities in curvature and inertia. *Mechanism and Machine theory*, 44(1):272–288, 2009.

[14] Michel Lalanne and Guy Ferraris. *Rotordynamics prediction in engineering*. Wiley, 1998.