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Use of permutation in determining a particular term of a determinant

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Abstract
The use of the idea of permutation in evaluating a determinant generates the terms of a determinant in a sequential order and hence can be used further in the determination of any particular term of the determinant at an instant. The way in which the terms of a determinant evolve is a matter of keen interest with which we deal with herein.

Keywords

Introduction and Main Results

The origin of matrices and determinants owes to the techniques used in solving a system of linear equations.[[1] and [2]]. The dictionary meaning of a determinant $|A|$ of a square matrix $A = (a_{ij})$ of order n as a square array of quantities called elements, kept within vertical bars, that symbolizes the sum of certain product of these elements. Each of the product in the sum contains a unique element from each row and a unique element from each column and hence contains the product of n elements. When the elements in the product are arranged sequentially on the basis of rows, the idea of arranging n elements of n columns solely play the deterministic role in determining a term or a product. If we write the elements column wise, a symbol (out of n symbols $1,2,3,\dots,n$) for column is fixed for each column; consequently, the $(n-1)!$ Ways of arranging the remaining symbols determine the $(n-1)!$ terms of a column. Hence forth, a totality of $n(n-1)! = n!$ terms results for n columns. The idea of moving column wise in succession or equivalently the value of j by 1 helps to determine the corresponding terms of the immediate next column.

We now state and prove our main result.

Theorem 1. *The determinant of a square matrix $A = (a_{ij})$ of order n is given by:*

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = \sum_{i=1}^{(n-1)!} \sum_{j=1}^{n} e_{ij}^1 e_{ij}^2 \dots e_{ij}^n$$

where $e_{ij}^k \in \{1,2,3,\dots,n\}, 1 \le k \le n$, such that $e_{il}^1 = l$ for all i written vertically as the first column and for $1 < k \le n$, e_{ij}^k is one among the remaining $(n-(k-1))$ symbols (positive integers) each written $(n-k)!$ times following $e_{ij}^{(k-1)}$ correspondingly, in the respective order $1,2,3,\dots,n$ of their occurrence. For $j > 1$ the elements e_{ij}^k are obtained by adding 1 to the corresponding elements of the preceding column and the sign is prefixed as mentioned before.

Proof. We use principle of mathematical induction to prove the theorem. To begin with we have for $n = 1$

$$\sum_{i=1}^{1} \sum_{j=1}^{1} e_{ij}^1 = \sum_{i=1}^{1} e_{i1}^1 = 1 = a_{11}$$

Hence the theorem holds for $n = 1$.

For $n = 2$, we have, $\sum_{i=1}^{1!} \sum_{j=1}^{2} e_{ij}^1 e_{ij}^2 = \sum_{i=1}^{1} (e_{i1}^1 e_{i1}^2 + e_{i2}^1 e_{i2}^2) = 12 - 21 = a_{11}a_{22} - a_{12}a_{21}$. This shows that the theorem holds for $n = 2$.

Now to prove the theorem by induction, let us assume that the theorem holds for a positive integer $n = m$. Then,

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \dots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{vmatrix} = \sum_{i=1}^{(m-1)!} \sum_{j=1}^{m} e^1_{ij} e^2_{ij} \dots e^m_{ij}$$

The signs being prefixed as stated in the theorem by multiplying each term by $(-1)^{k-1}$ where k is the number of symbols out of $1, 2, 3, \cdots, n$ mapped to different symbols. Now it remains to show that the theorem holds for $n = m+1$ assuming it to be true for $n = m$. When the symbolic form for the expression of the determinant of order m is expanded we get m! arrangements of the m symbols $1, 2, 3, \cdots, m$ in the form of m columns each containing $(m-1)!$ terms (rows) Further in the first column $e^1_{i1} = 1$ for all i which occurs $(m-1)!$ times followed by the remaining $m-1$ symbols $2, 3, 4, \cdots, m$ each occurring $(m-2)!$ times in their respective order so as to constitute e^2_{ij} and so forth; e^m_{ij} will then be the last remaining symbol and occurs $(m-m)! = 1$ time accordingly. Shifting (adding 1) each symbols one unit rightwards changes $1, 2, 3, \cdots, m$ to $2, 3, 4, \cdots, m, 1$ respectively as the symbol next to m being 1 again, these symbols so obtained constitute the second column. The shifting can be done only up to $m-1$ steps after which we get back the first column again. This is the reason behind adding $'1'$ to each symbol of a column to get the corresponding elements of the next column. It is clear from above that the idea of permutations of $'m'$ symbols $1, 2, 3, \cdots, m$ is a basis of the above representation. Since the permutations of objects holds for all $n \in \mathbb{N}$; the above representation must hold for $n = m+1$, if it holds for $n = m$. Further the rule of pre-fixing the signs since holds for all k must also hold for $n = m+1$. This in other words means that

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1(m+1)} \\ a_{21} & a_{22} & \dots & a_{2(m+1)} \\ \dots \\ a_{(m+1)1} & a_{(m+1)2} & \dots & a_{(m+1)(m+1)} \end{vmatrix}$$

$$= \sum_{i=1}^{(m)!} \sum_{j=1}^{m+1} e^1_{ij} e^2_{ij} \dots e^{m+1}_{ij}$$

This proves that the theorem holds for $n = m+1$ if it holds for $n = m$. Hence by the principle of mathematical induction the theorem holds for all $n \in \mathbb{N}$. □

Lemma 2. *For any positive integer $m < n!$ the mth term of a determinant of order n is given by $t_m =^* s_n(n-k)$ where k denotes the column (of arrangements of elements) and the value of $s_n (0 \le s_n \le n-k)$ for $k = 1, 2, 3, \cdots, n$ denote the steps we move ahead to determine the symbols $e^1_{ij}, e^2_{ij}, \cdots, e^n_{ij}$ respectively constituting the term.*

Proof. According to the theorem for the expansion of determinant, the value of a determinant of order $'n'$ on $'n'$ symbols $1, 2, 3, \cdots, n$ of second suffix (i.e.j in a_{ij}) as represented by $e^k_{ij} (1 \le k \le n)$ is given by

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = \sum_{i=1}^{(n-1)!} \sum_{j=1}^{n} e^1_{ij} e^2_{ij} \dots e^n_{ij}$$

where i and j stand for the row (increasing vertically) and column (increasing horizontally) respectively of an element $a_i j$; such that $e^1_{ij} = j$ for all i. In particular $e^1_{il} = 1$ for all i written $(n-1)!$ times vertically in the first column. For $l < k \le n$; e^k_{il} is one among the remaining $(n-(k-1))$ symbols out of $1, 2, 3, \cdots, n$ each written $(n-k)!$ along the column corresponding to the symbol e^{k-1}_{i1} in the respective order $1, 2, 3, \cdots, n$ of their occurrence. For $j > 1$, the elements $e^k_{ij} (1 \le k \le n)$ are obtained by adding 1 (i.e. moving 1 unit right ward) to the corresponding elements of the preceding column. (The sign of each term after the +ve first is prefixed by multiplying it by $(-1)^{k-1}$ is the number of symbols out of $1, 2, 3, \cdots, n$ mapped to different symbols. In case $1, 2, 3, \cdots n$ form different disjoint groups, the rule is applicable separately for each group, the product of the signs being the sign of the term. For the last symbol $k = n$ and $(n-k)! = (n-n)! = 0! = 1$ (and hence sn has no meaning). So taking off the fixed (1) option for the last symbol, the number of options for the determination becomes $m-1$. Now divide $m-1$ by $(n-1)!$ to get a remainder r_1, which is divided by $(n-2)!$ to get a remainder r_2 which in turn is divided by (n-3)! to get a remainder r_3 and proceed so forth until we get r_{n-1} and note the quotient sn in each case. For $k = 1$, if $s_1 = 0$ the term lies in the first column. For

$s_n > 0$, the term lies in the column other than the 1st since by the theorem for expansion of determinant, terms of other column can be obtained by adding proper index (s_1) to each symbols of the corresponding terms of the first column. So we replace s_1 by 0 and write the corresponding terms of the first column. The non-zero value of S_1 is then applied to move each symbol by 's_1' units to get the required column. For $k = 2, 3, \cdots, n-1$. the corresponding values of $'s'_n$ suggest us to move forward within the column to locate the $2nd, 3rd, \cdots, (n-1)th$ symbols respectively. Hence writing all the symbols determines the term.

Illustrations

We determine the 3rd, 8th and 23th terms of a 4×4 determinant. We also determine the 7th, 23rd, 74th and 117th term of a determinant of order 5.

We have for a determinant of order 4

$$(4-1)! = 6, (4-2)! = 2, (4-3)! = 1, (4-4)! = 0! = 1$$

Now,

(a) $3 = 0 \times 6 + 1 \times 2 + 0 \times 1 + 1$ $S_1 = 0$, the term lies in the 1st column. So, $e'_{ij} = e'_{il} = 1$. For e^2_{ij}, symbols available in order are $2, 3, 4$ $S_2 = 1$, we move 1 step ahead from 2 which is 3 $e^2_{ij} = 3$. Now symbols left in order are $2, 4$ $S_3 = 0$, we do not move ahead of 2 for e^3_{ij} $e^3_{ij} = 2$. Hence e_{4ij} is the remaining symbol 4. Therefore, $3 = 0 \times 6 + 1 \times 2 + 0 \times 1 + 1 = 1324 = a_{11}a_{23}a_{32}a_{44}$ (in magnitude) For sign; considering mapping from $\{1, 2, 3, 4\}$ to $\{1, 3, 2, 4\}$ only 2 and 3 are mapped to different symbols $(-1)^{2-1} = -$ sign is prefixed. Hence 3rd term = $a_{11}a_{23}a_{32}a_{44}$

(b) $8 = 1 \times 6 + 0 \times 2 + 1 \times 1 + 1$ Replacing the coeff. of $(n-1)! = 6$ by 0 the corresponding term of first column is given by $0 \times 6 + 0 \times 2 + 1 \times 1 + 1$ which is 1243 $S_1 = 1$ we add 1 to each symbol to get 23144. Since $1, 2, 3$ are mapped to diff. symbols $k = 3$, therefore, + sign is prefixed. Therefore, 8th term is 2314 i.e. $a_{11}a_{23}a_{32}a_{44}$

(c) $23 = 3 \times 6 + 2 \times 2 + 0 \times 1 + 1$
Replacing $s_1 = 3$ by 0 yields $0 \times 6 + 2 \times 2 + 0 \times 1 + 1 = 1423$. So moving each symbols 3 units forward yields 4312 (all are mapped to diff. symbols i.e. $k = 4$) 23rd term = $+4312 = a_{14}a_{23}a_{31}a_{42}$.

Now for a determinant of order 5 $(5-1)! = 24, (5-2)! = 6, (5-3)! = 2, (5-4)! = 1, (5-5)! = 0! = 1$ Now,

(a) $7 = 0 \times 24 + 1 \times 6 + 0 \times 2 + 0 \times 1 + 1$ for the corresponding element is $+13245 = a_{11}a_{23}a_{32}a_{44}a_{55}$

(b) $23 = 0 \times 24 + 3 \times 6 + 2 \times 2 + 0 \times 1 + 1$ The corresponding term is $-15423 = -a_{11}a_{25}a_{34}a_{42}a_{53}$

(c) $74 = 3 \times 24 + 0 \times 6 + 0 \times 2 + 1 \times 1 + 1$ The corresponding term is (12354) moved by 3 units $= -45132 = -a_{14}a_{25}a_{31}a_{43}a_{52}$

(d) $117 = 4 \times 24 + 3 \times 6 + 1 \times 2 + 0 \times 1 + 1$

The corresponding term is (15324) moved 4 units to the right = $+54213 = a_{15}a_{24}a_{32}a_{41}a_{53}$.

□

References

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