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Coupled Lateral and Torsional Vibration of a Rigid Disk Attached to a Flexible Shaft

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Abstract

Various rotational mechanical equipment consists of disk attached to a shaft. These undergo torsion and bending in operation. The coupled effect of lateral and torsional vibration is a common reason for failure of system which creates a necessity for it to be studied so that it may help curb the potential failures. Coupled torsional-transverse vibration for a rigid disk attached to the middle of a flexible shaft system is studied. The equations of motion are established by applying the Hamilton equation to the energy expressions and by applying the kinematic constraints. Expressing the displacement as perturbation series separates the governing equation into different orders. This showed interaction of torsional and transverse term at second and higher order. To better understand the interaction, rigid disk attached to a simply supported uniform shaft is considered and its equation of motion is solved using eigenfunction expansion. This response showed that resonance occurs if the critical speed is half of the torsional natural frequency. A numerical example is also considered to determine the characteristics of torsional vibration as a result of this interaction.

Keywords

Coupled Vibration, Perturbation, Eigenfunction Expansion

1. Introduction

Shaft disk system is common in modern industrial and power generation facilities. Pelton turbine is a common example of this type of system. The response of these shaft disk system is of tremendous importance and these type of systems has been subjected to many different studies which focuses on different aspects of the system such as free vibration , forced vibration and at different orientations and end conditions[1][2][3][4]. This paper focuses on the interaction between transverse and torsional vibration of a shaft disk system.

Interaction between torsional and transverse deformation in a rotating systems has been subjected to some study in the past. Tondl first studied the coupled effect of lateral and torsional vibrations of a turbo-generator rotor[5]. Cohen and Porat investigated coupled torsional and transverse vibration of unbalanced rotor[6]. Nataraj found that this interaction takes place at second order[7]. Al-Bedoor modelled the coupling by treating torsional angle as an individual motion[8]. Shen found that external force excite torsional vibration and external moment excite lateral vibration [9]. While coupled vibration focusing on different aspects of a shaft system are studied, there is an alarming lack of research on coupled vibration of shaft disk system.

This research work will develop equation of motion to study the coupling of lateral and torsional vibration and develop analytical solution of coupled lateral and torsional vibration when a rigid disk is attached to a simply supported shaft.

2. Methodology

Initially, the velocity expressions for the system is determined which is then used to obtain the energy and work expressions. Then, these expressions for work and energy are substituted into generalized Hamilton Principle and after applying kinematic constraints, partial differential equations associated with each coordinate is obtained. Separating the terms for different order using perturbation series resulted in equation of motion at different orders to study the order of interaction between torsional and lateral vibration. To gain more insight into this interaction, expression for torsional vibration of second order is solved in a case of uniform system.

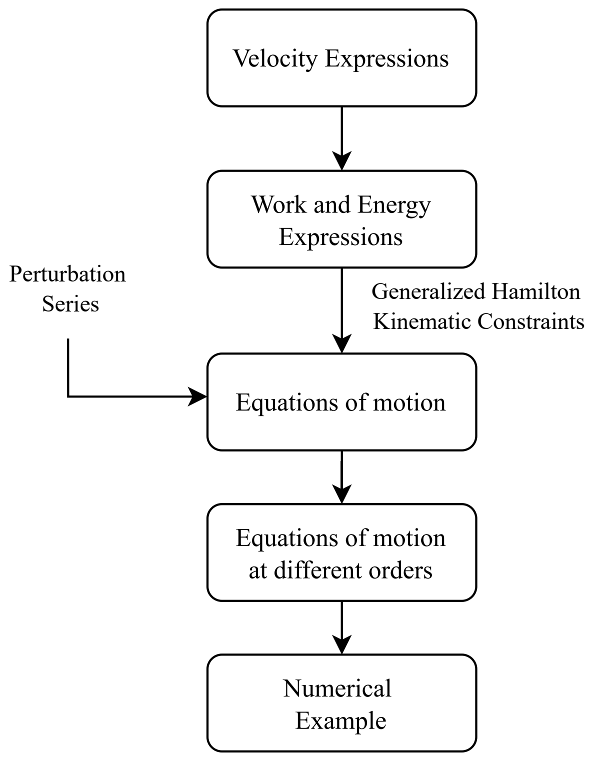

Figure 1: Methodology

3. Mathematical Modelling

3.1 Assumptions

The following assumptions are made:

1. Disk is assumed to be rigid
2. Displacement in axial direction is neglected
3. Damping in bearing is neglected
4. Torsional and transverse deformations are small and of same order

3.2 Coordinate System

Let $OXYZ$ be a fixed coordinate system which is considered to be an inertial frame of reference. Let $o_1x_1y_1z_1$ be cartesian frame of reference which is translating but not rotating with refrence to $OXYZ$. Let $oxyz$ be a body fixed coordinate. If the two systems are initially coincident, a series of rotataion performed in proper sequence of ψ about z, θ about y, and ϕ about x is sufficient to allow the $oxyz$ system to reach any orientation. This 3-2-1 matrix transformation is given as:

$$\begin{bmatrix} x & y & z \end{bmatrix}^T = T_{321} \begin{bmatrix} x_1 & y_1 & z_1 \end{bmatrix}^T \tag{1}$$

where T_{321} is given as,

$$\begin{bmatrix} c\psi c\theta & s\psi c\theta & -s\theta \\ -s\psi c\phi + c\psi s\theta s\phi & c\psi c\phi + s\psi s\theta s\phi & c\theta s\phi \\ s\psi s\phi + c\psi s\theta c\phi & -c\psi s\phi + s\psi s\theta c\phi & c\theta c\phi \end{bmatrix}$$

Here c and s represent cos and sin respectively.

3.3 Velocity Expressions

Angular velocity ω of the xyz system is the vector sum of time derivatives of Euler angles[10]:

$$\omega = \dot{\psi} + \dot{\theta} + \dot{\phi} \tag{2}$$

This can be resolved onto each axes of xyz system into [11]:

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \dot{\phi} - \dot{\psi}\sin\theta \\ \dot{\theta}\cos\phi + \dot{\psi}\cos\theta\sin\phi \\ -\dot{\theta}\sin\phi + \dot{\psi}\cos\theta\sin\phi \end{bmatrix} \tag{3}$$

Again, since it is assumed that the axial velocity ($\dot{u}$) is zero, the velocity in xyz system is given by:

$$V = \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} = \begin{bmatrix} T_{321} \end{bmatrix} \begin{bmatrix} 0 \\ \dot{v} \\ \dot{w} \end{bmatrix} + \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \times \begin{bmatrix} 0 \\ e\cos\beta \\ e\sin\beta \end{bmatrix} =$$
$$\begin{bmatrix} \dot{v}\sin\psi\cos\theta - \dot{w}\sin\theta + e\dot{\theta}\sin(\phi+\beta) \\ -e\dot{\psi}\cos(\phi+\beta) \\ \\ \dot{v}(\cos\psi\cos\theta + \sin\psi\sin\theta\sin\phi) - \dot{w}\cos\theta\sin\phi \\ +e(\dot{\phi} - \dot{\psi}\sin\theta)\sin\beta \\ \\ \dot{v}(-\cos\psi\sin\phi + \sin\psi\sin\theta\cos\phi) - \dot{w}\cos\theta\cos\phi \\ +e(\dot{\phi} - \dot{\psi}\sin\theta)\cos\beta \end{bmatrix} \tag{4}$$

Here, e and β denotes the unbalance eccentricity and unbalance phase respectively.

3.4 Energy Expressions

3.4.1 Shaft

As, the shaft is considered flexible, it will be characterised by both kinetic and potential energy. The kinetic energy of the shaft is given by:

$$T = \frac{1}{2}\rho A \int_0^L V^2\,dx + \frac{1}{2}I_p \int_0^L {\omega_x}^2\,dx + \frac{1}{2}I_d \int_0^L (\omega_y + \omega_z)^2\,dx \tag{5}$$

Also, the potential energy of the shaft is given by:

$$U = \frac{EI}{2}\int_0^L \left[\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial^2 v}{\partial x^2}\right)^2\right] dx + \frac{GJ}{2}\int_0^L \left(\frac{\partial \phi}{\partial x}\right)^2 dx \tag{6}$$

3.4.2 Disk

As, the disk is considered to be uniform and rigid, it will be characterised by only the potential energy. The kinetic energy of the disk is given by:

$$T_d = \frac{1}{2}M_d\left(\dot{v}^2 + \dot{w}^2\right)\Big|_{x=\frac{L}{2}} + \frac{1}{2}I_{pd}{\omega_x}^2\Big|_{x=\frac{L}{2}} + \frac{1}{2}I_{dd}\left({\omega_y}^2 + {\omega_z}^2\right)\Big|_{x=\frac{L}{2}} \tag{7}$$

3.4.3 Virtual Work

Work done is given by:

$$W = F_d v|_{x=\frac{L}{2}} + T_d\frac{\partial \phi}{\partial x}\Big|_{x=\frac{L}{2}} - T_d\frac{\partial \psi}{\partial x}\sin\theta\Big|_{x=\frac{L}{2}} \tag{8}$$

3.5 Kinematic Constraints

Since the angular displacements θ and ψ are considered small, we can express θ and ψ as[4]:

$$\theta = -\frac{\partial w}{\partial x} \tag{9}$$

$$\psi = \frac{\partial v}{\partial x} \tag{10}$$

3.6 Equation of Motion

Applying generalized Hamilton's principle to the energy expressions gives the equation of motion and associated boundary condition.

$$\int_{t_1}^{t_2}\left[\delta\left(T - U\right) + \delta W\right] dt = 0 \tag{11}$$

Neglecting higher order terms, the equation of motion is:

$$\begin{aligned}(EIv")" + m\ddot{v} - \left(I_d\ddot{v}'\right)' - \left(I_p\ddot{\phi}w'\right)' - \left(I_p\dot{\phi}\dot{w}'\right)' \\ = \frac{\partial}{\partial t}\left(me\dot{\phi}\sin(\phi+\beta)\right) - M_d\ddot{v}\delta\left(x-\frac{L}{2}\right) \\ + \left(I_{dd}\ddot{v}'\right)'\delta\left(x-\frac{L}{2}\right) + \left(I_{pd}\ddot{\phi}w'\right)'\delta\left(x-\frac{L}{2}\right) \\ + \left(I_{pd}\dot{\phi}\dot{w}'\right)'\delta\left(x-\frac{L}{2}\right) - \left(\tau_d w'\right)'\delta\left(x-\frac{L}{2}\right) \\ + F_d\delta\left(x-\frac{L}{2}\right)\end{aligned} \tag{12}$$

$$\begin{aligned}(EIw")" + m\ddot{w} - \left(I_d\ddot{w}'\right)' + \left(I_p\dot{\phi}\dot{v}'\right)' = \\ -\frac{\partial}{\partial t}\left(me\dot{\phi}\cos(\phi+\beta)\right) - M_d\ddot{w}\delta\left(x-\frac{L}{2}\right) \\ + \left(I_{dd}\ddot{w}'\right)'\delta\left(x-\frac{L}{2}\right) - \left(I_{pd}\dot{\phi}\dot{v}'\right)'\delta\left(x-\frac{L}{2}\right)\end{aligned} \tag{13}$$

$$\begin{aligned}\left(GJ\phi'\right)' - I_p\ddot{\phi} - me^2\ddot{\phi} - I_p\frac{\partial}{\partial t}\left(\dot{v}'w'\right) \\ = -me\left[\ddot{v}\sin\left(\beta+\phi\right) - \ddot{w}\sin\left(\beta+\phi\right)\right] \\ + I_{pd}\ddot{\phi}\delta\left(x-\frac{L}{2}\right) + I_{pd}\frac{\partial}{\partial t}\left(\dot{v}'w'\right)\delta\left(x-\frac{L}{2}\right) \\ - \tau_d\delta\left(x-\frac{L}{2}\right)\end{aligned} \tag{14}$$

3.7 Perturbation Series

To better understand the extent of interaction, assuming small deformation and constant angular velocity of the rotor, the generalized coordinates can be expressed in the form of perturbation series as:

$$v = \varepsilon v_1 + \varepsilon^2 v_2 + \cdots + \varepsilon^n v_n + \cdots \tag{15}$$

$$w = \varepsilon w_1 + \varepsilon^2 w_2 + \cdots + \varepsilon^n v_n + \cdots \tag{16}$$

$$\phi = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots + \varepsilon^n\phi_n + \cdots \tag{17}$$

$$\tau_d = \varepsilon\tau_1 + \varepsilon^2\tau_2 + \cdots + \varepsilon^n\tau_n + \cdots \tag{18}$$

Here, $\dot{\phi}_0 = \Omega$ is the constant rotational speed of the shaft. Removing zeroth order term from the torque series is consistent with the assumption of constant rotational speed.[7]

Using equation 17 and by Taylor expansion:

$$\sin(\phi+\beta) = \sin(\phi_0+\beta) + \varepsilon\phi_1\cos(\phi_0+\beta) + \cdots \tag{19}$$

$$\cos(\phi+\beta) = \cos(\phi_0+\beta) - \varepsilon\phi_1\sin(\phi_0+\beta) + \cdots \tag{20}$$

3.8 Equation of Motion at different orders

3.8.1 First Order

At $O(\varepsilon)$, the equation of motion is:

$$\begin{aligned}(EIv_1'')'' + m\ddot{v}_1 - \left(I_d\ddot{v}_1'\right)' - \left(I_p\Omega\dot{w}_1'\right)' =\\ \frac{\partial}{\partial t}\left(me_1\Omega\sin(\phi_0+\beta)\right) - M_d\ddot{v}_1\delta\left(x-\frac{L}{2}\right)\\ + \left(I_{pd}\Omega\dot{w}_1'\right)'\delta\left(x-\frac{L}{2}\right) + \left(I_{dd}\ddot{v}_1'\right)'\delta\left(x-\frac{L}{2}\right)\\ + (F_1)\,\delta\left(x-\frac{L}{2}\right)\end{aligned} \tag{21}$$

$$\begin{aligned}(EIw_1'')'' + m\ddot{w}_1 - \left(I_d\ddot{w}_1'\right)' + \left(I_p\Omega\dot{v}_1'\right)' =\\ -\frac{\partial}{\partial t}\left(me_1\Omega\cos(\phi_0+\beta)\right) - M_d\ddot{w}_1\delta\left(x-\frac{L}{2}\right)\\ + \left(I_{dd}\ddot{w}_1'\right)'\delta\left(x-\frac{L}{2}\right) - \left(I_{pd}\Omega\dot{v}_1'\right)'\delta\left(x-\frac{L}{2}\right)\end{aligned} \tag{22}$$

$$\begin{aligned}\left(GJ\phi_1'\right)' - I_p\ddot{\phi}_1 + \tau_1\delta\left(x-\frac{L}{2}\right) =\\ I_{pd}\ddot{\phi}_1\delta\left(x-\frac{L}{2}\right)\end{aligned} \tag{23}$$

It is clear that the transverse and torsional terms are completely decoupled in this case as there is no torsional vibration terms in transverse vibration equation and there is no transverse vibration term in torsional vibration equation. This is to say that, torsional vibration cannot induce torsional vibration and vice versa at first order.

3.8.2 Second Order

At $O(\varepsilon^2)$, the equation of motion is:

$$\begin{aligned}(EIv_2'')'' + m\ddot{v}_2 - \left(I_d\ddot{v}_2'\right)' - \left(I_p\Omega\dot{w}_2'\right)'\\ = \left(I_p\ddot{\phi}_1 w_1'\right)' + \left(I_p\dot{\phi}_1\dot{w}_1'\right)' + \frac{\partial}{\partial t}\left(me_2\Omega\sin(\phi_0+\beta)\right)\\ + \frac{\partial}{\partial t}\left(me_1\Omega\phi_1\cos(\phi_0+\beta)\right) + \frac{\partial}{\partial t}\left(me_1\dot{\phi}_1\sin(\phi_0+\beta)\right)\\ -M_d\ddot{v}_2\delta\left(x-\frac{L}{2}\right) + \left(I_{dd}\ddot{v}_2'\right)'\delta\left(x-\frac{L}{2}\right)\\ + \left(I_{pd}\ddot{\phi}_1 w_1'\right)'\delta\left(x-\frac{L}{2}\right) + \left(I_{pd}\Omega\dot{w}_2'\right)'\delta\left(x-\frac{L}{2}\right)\\ + \left(I_{pd}\dot{\phi}_1\dot{w}_1'\right)'\delta\left(x-\frac{L}{2}\right) - \left(\tau_1 w_1'\right)'\delta\left(x-\frac{L}{2}\right)\\ + (F_2)\,\delta\left(x-\frac{L}{2}\right)\end{aligned} \tag{24}$$

$$\begin{aligned}(EIw_2'')'' + m\ddot{w}_2 - \left(I_d\ddot{w}_2'\right)' + \left(I_p\Omega\dot{v}_2'\right)'\\ = -\left(I_p\dot{\phi}_1\dot{v}_1'\right)' - \frac{\partial}{\partial t}\left(me_2\Omega\cos(\phi_0+\beta)\right)\\ + \frac{\partial}{\partial t}\left(me_1\Omega\phi_1\sin(\phi_0+\beta)\right) - \frac{\partial}{\partial t}\left(me_1\dot{\phi}_1\cos(\phi_0+\beta)\right)\\ -M_d\ddot{w}_2\delta\left(x-\frac{L}{2}\right) + \left(I_{dd}\ddot{w}_2'\right)'\delta\left(x-\frac{L}{2}\right)\\ - \left(I_{pd}\Omega\dot{v}_2'\right)'\delta\left(x-\frac{L}{2}\right) - \left(I_{pd}\dot{\phi}_1\dot{v}_1'\right)'\delta\left(x-\frac{L}{2}\right)\end{aligned} \tag{25}$$

$$\begin{aligned}\left(GJ\phi_2'\right)' - I_p\ddot{\phi}_2 = I_p\frac{\partial}{\partial t}\left(\dot{v}_1' w_1'\right) - \tau_2\delta\left(x-\frac{L}{2}\right)\\ -me_1\left[\ddot{v}_1\sin(\theta_0+\beta) - \ddot{w}_1\cos(\theta_0+\beta)\right]\\ +I_{pd}\ddot{\phi}_2\delta\left(x-\frac{L}{2}\right) + I_{pd}\frac{\partial}{\partial t}\left(\dot{v}_1' w_1'\right)\delta\left(x-\frac{L}{2}\right)\end{aligned} \tag{26}$$

It is clearly seen that torsional vibration appears as a forcing term in the equations for transverse vibrations and transverse vibration appear as a forcing term in the equation for torsional vibration. This suggests that interaction between transverse and torsional vibrations occur at second and higher orders.

4. Solution

To better understand the interaction, rigid disk attached to a simply supported uniform shaft with constant angular speed Ω is considered. Also, it is assumed that there is no torque and force fluctuations.

4.1 First Order

For this case, it can be clearly seen from equations (21), (22) and (23) that while the transverse motion are coupled with one another, torsional motion is independent of transverse motion and can be solved easily to yield eigenmodes.

To solve equation of motion of $O(\varepsilon)$, using assumed mode method, the transverse displacement variables can be expressed as:

$$v_1(x,t) = \sum_{n=1}^{\infty} V_n(t) \sin\left(\frac{n\pi x}{l}\right) \tag{27}$$

$$w_1(x,t) = \sum_{n=1}^{\infty} W_n(t) \sin\left(\frac{n\pi x}{l}\right) \tag{28}$$

$$\phi_1(x,t) = \sum_{n=1}^{\infty} \Phi_n(t) \sin\left(\frac{n\pi x}{l}\right) \tag{29}$$

Using orthogonality condition, equation of motion for lateral vibration reduces from a coupled system of PDE to a coupled system of ODE as:

$$M_n \ddot{V}_n(t) + C_n \dot{W}(t) + K_n(t) = 0 \tag{30}$$

$$M_n \ddot{W}_n(t) - C_n \dot{V}(t) + K_n(t) = 0 \tag{31}$$

where,

$$M_n = \frac{mL}{2} + I_d \frac{n^2\pi^2}{l^2}\frac{l}{2} + M_d \sin^2\left(\frac{n\pi}{2}\right) + I_{dd}\frac{n^2\pi^2}{l^2}\sin^2\left(\frac{n\pi}{2}\right) \tag{32}$$

$$C_n = \left[I_p \frac{n^2\pi^2}{l^2}\frac{l}{2} + I_{pd}\frac{n^2\pi^2}{l^2}\sin^2\left(\frac{n\pi}{2}\right)\right]\Omega = C_{n_c}\Omega \tag{33}$$

$$K_n = EI\frac{n^4\pi^4}{l^4}\frac{l}{2} \tag{34}$$

The above ODE is satisfied by:

$$V_n(t) = A_{vn}\cos(\Omega_n t + \beta) \tag{35}$$

$$W_n(t) = A_{wn}\sin(\Omega_n t + \beta) \tag{36}$$

From this, the characteristics equation of the system is obtained as:

$$M_n^2 {\Omega_n}^4 - \left({C_n}^2 + 2K_n M_n\right){\Omega_n}^2 + K_n^2 = 0 \tag{37}$$

Here, Ω_n gives the natural frequencies for backward and forward whirl of the system at nth mode for a spin speed Ω and is given by:

$$\Omega_n^2 = \frac{C_{n_c}^2\Omega^2 + 2K_nM_n \pm C_{n_c}\Omega\sqrt{\left(C_{n_c}^2\Omega^2 + 4K_nM_n\right)}}{2M_n^2} \tag{38}$$

Similarly using orthogonality condition, the equation for torsional vibration reduces to a ordinary differential equation as:

$$\left[I_p\frac{l}{2} + I_{pd}\sin^2\left(\frac{n\pi}{2}\right)\right]\ddot{\Phi}_n(t) + \left[\frac{n^2\pi^2}{l^2}GJ\right]\Phi_n(t) = 0 \tag{39}$$

From equation (39), it can be clearly seen that the torsional natural frequency of nth mode is given as:

$$\delta_n = \sqrt{\frac{\frac{n^2\pi^2}{l^2}GJ}{I_p\frac{l}{2} + I_{pd}\sin^2\left(\frac{n\pi}{2}\right)}} \tag{40}$$

4.2 Second Order

A equivalent formulation of $O(\varepsilon)$ transverse displacement variables is:

$$v_1(x,t) = \sum_{n=1}^{\infty} A_{vn}\sin\left(\frac{n\pi x}{l}\right)\cos(\Omega_n t + \beta) \tag{41}$$

$$w_1(x,t) = \sum_{m=1}^{\infty} A_{vm}\sin\left(\frac{m\pi x}{l}\right)\sin(\Omega_n t + \beta) \tag{42}$$

where A_{vn} and A_{wm} are the amplitude of response at each mode.

Substituting equations (41) and (42) in equation (26) yields the equation for torsional motion at $O(\varepsilon^2)$ for this case:

$$GJ\phi_2" - I_p\ddot{\phi}_2 - I_{pd}\ddot{\phi}_2\delta\left(x - \frac{L}{2}\right)$$
$$= -I_p\Omega_n{}^2\left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_m A_n \frac{nm\pi^2}{l^2}\cos\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\right]\sin 2(\Omega_n t + \beta)$$
$$-I_{pd}\Omega_n{}^2\left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_m A_n \frac{nm\pi^2}{l^2}\cos\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\right]\sin 2(\Omega_n t + \beta)\delta\left(x - \frac{L}{2}\right) \quad (43)$$

It is natural to assume the solution to be of the form:

$$\phi_2(x,t) = \sum_{k=1}^{\infty} A_k \sin\left(\frac{k\pi x}{l}\right)\sin 2(\Omega_n t + \beta) \quad (44)$$

Replacing equation (44) into equation (43), we get the Residual. Using Galerkin's weighted residual, we obtain the numerical value of A_k

$$A_k = \frac{1}{4}\frac{F_k}{\left[-1 + \left(\frac{\delta_k}{2\Omega_n}\right)^2\right]} \quad (45)$$

where the kth modal generalized force due to flexural motion (F_k) is given by:

$$F_k = \frac{1}{\left(I_p\frac{l}{2} + I_{pd}\sin^2\frac{k\pi}{2}\right)}\left[I_{pd}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{wm}A_{vn}\frac{nm\pi^2}{l^2}\cos\frac{m\pi}{2}\cos\frac{n\pi}{2}\sin\frac{k\pi}{2} + I_p\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{wm}A_{vn}\frac{nm\pi^2}{l^2}\times\int_0^L \cos\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\sin\frac{k\pi x}{l}\,dx\right] \quad (46)$$

This response shows that flexural motion of frequency Ω_n induces torsional motion with twice the frequency. Moreover, it is obvious that resonance can occur if the critical speed (Ω_n) is close to half of torsional natural frequency (δ_k). As a validation, removing the disk from the model, equations (43) to (46) reduces to the findings as obtained by [7].

5. Numerical Example

As a numerical example solution is carried out for the first mode of a shaft disk system as illustrated in the figure.

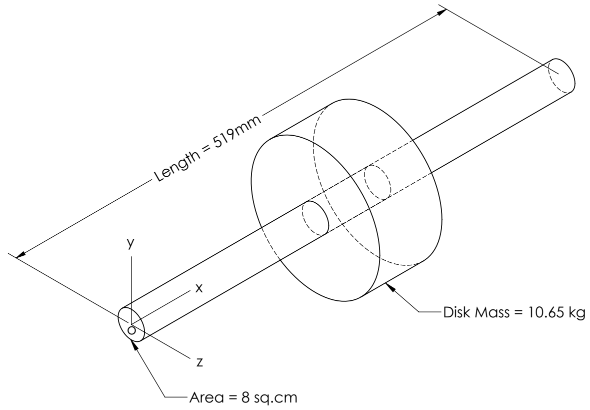

Figure 2: Shaft Disk System

The following parameters based on a Pelton turbine unit set-up with capacity of 2kW rated at 1500rpm are considered:

Parameters	Value
Mass of Disk (M_d)	10.65 kg
Polar MMOI of Shaft (I_p)	4.16 $kgcm^2$
Diametral MMOI of Shaft (I_d)	734.62 $kgcm^2$
Polar MMOI of Disk (I_{pd})	334.00 $kgcm^2$
Diametral MMOI of Disk (I_{dd})	216.80 $kgcm^2$
Length of Shaft (L)	519.00 mm
Density of Shaft (ρ)	7860.00 kg/m^3
Cross Section of Shaft (A)	8.00 cm^2
Polar AMOI of Shaft (J)	101859.16 mm^4
Diametral AMOI of Shaft (I)	50929.58 mm^4
Young's Modulus of Shaft (E)	202.00 GPa
Shear Modulus of Shaft (G)	84.00 GPa

Table 1: Model Parameters

5.1 First Order

Using the model parameters, and considering only the first mode, equations (32) to (34) reduces to:

$$M_1 = \frac{mL}{2} + I_d\frac{\pi^2}{l^2}\frac{l}{2} + M_d + I_{dd}\frac{\pi^2}{l^2} = 12.295 \quad (47)$$

$$C_1 = I_p\Omega\frac{\pi^2}{l^2}\frac{l}{2} + I_{pd}\Omega\frac{\pi^2}{l^2} = 1.228\Omega = C_{1_c}\Omega \quad (48)$$

$$K_1 = EI\frac{\pi^4}{l^4}\frac{l}{2} = 3584172.50 \tag{49}$$

Replacing these values in equation (37) we obtain the critical speeds for first mode as as a function of spin speed:

$${\Omega_1}^2 = \frac{C_{1_c}^2\Omega^2 + 2K_1M_1 - C_{1_c}\Omega\sqrt{\left(C_{1_c}^2\Omega^2 + 4K_1M_1\right)}}{2{M_1}^2} \tag{50}$$

As a validation, evaluation of critical speed at static case gives the value of 539.921 rad/s, which is close to the value of 544.78 rad/s as obtained by [12] for a non-identical model of the same pelton turbine unit. And, forward and backward whirl at 1500RPM gives the value of 547.822 rad/s and 532.134 rad/s respectively, which is close to the values obtained by [4].

5.2 Second Order

As only the first mode of torsional displacement is being considered, equation (44) reduces to:

$$\phi_2(x,t) = \frac{F_1 \sin\left(\frac{\pi x}{l}\right)\sin 2\left(\Omega_1 t + \beta\right)}{4\left[-1 + \left(\frac{\delta_1}{2\Omega_1}\right)^2\right]} \tag{51}$$

From equation (40), first mode torsional natural frequency (δ_1) is obtained and the first mode critical speed(Ω_1) as a function of spin speed can be obtained from (51). Amplitude of second order torsional response per unit first modal generalized force(F_1) can be presented as shown in figure 3:

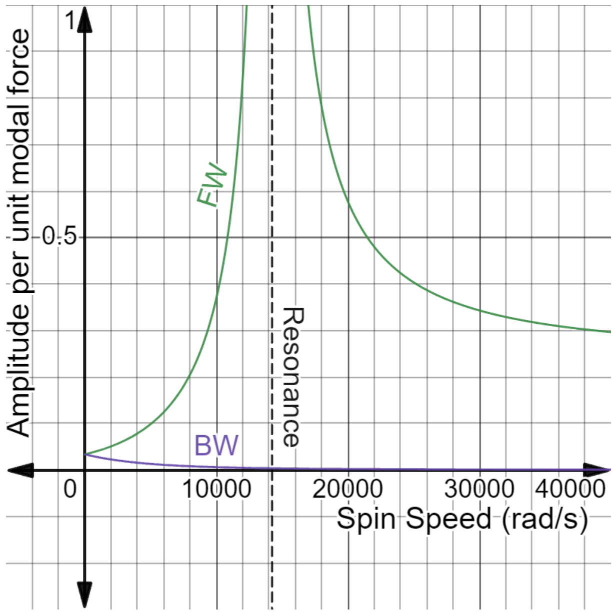

Figure 3: Amplitude of torsional response as a function of spin speed for first mode

If the spin speed is chosen to be 1500RPM, then the second order torsional displacement per unit first modal generalized force can be visualized as shown in figure 4:

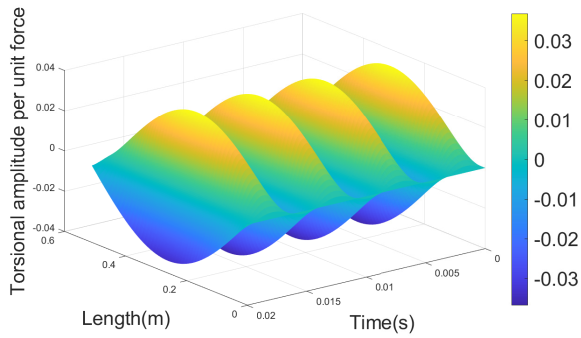

Figure 4: Second order torsional response as a function of time and length for first mode

It can be seen that the response is a sinusoid in both spatial and temporal direction and the deflection is maximum at the center with an amplitude to force ratio of 0.036

6. Conclusions

Equation of motion for a rigid disk flexible shaft system with the disk at the middle is obtained using generalized Hamilton's condition. Subjecting these to perturbation approach separated the equation of motion to different order. While the first order torsional motion showed free vibration, second order showed a forced vibration, wherein the first order flexural terms acted as the forcing term. Moreover, the frequency of torsional motion is found to be double the flexural frequency. This shows that the interaction occurs at second and higher order. Response of second order torsional vibration is obtained which showed that resonance occurs if the critical speed is half of the torsional natural frequency.

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