

Three-Dimensional Elastic Analysis of Flexible Pavement under Static Vehicular Load

Bijay Ban ^a, Jagat K. Shrestha ^b, Rojee Pradhananga ^c, Kshitij C. Shrestha ^d

^{a, b, c, d} Department of Civil Engineering, Pulchowk Campus, IOE, Tribhuvan University, Nepal

Corresponding Email: ^a 075msste001.bijay@pcampus.edu.np, ^b jagatshrestha@ioe.edu.np,
^c rojee.pradhananga@pcampus.edu.np, ^d kshitij.shrestha@pcampus.edu.np

Abstract

Structural analysis of pavement is essential to investigate effect of various factors affecting pavement condition. This paper presents a three-dimensional finite element model suitable for structural analysis of flexible pavements in Nepal. The pavement model is developed using finite element software ANSYS and can simulate pavement responses under static vehicular load. The vehicular loading and material properties parameters are defined based on Department of Roads Nepal's guideline for design of flexible pavements. The pavement model is validated with the classic theoretical responses formulated for single layer pavement system. The validated model is further utilized to investigate the pavement responses of a three layered pavement system. The pavement responses are simulated and compared for two interface conditions, bonded contact and friction contact between the pavement layers. The test results shows decrease in pavement responses over the depth of the pavement and increase in pavement responses when the interface layers are not perfectly bonded.

Keywords

Finite Element Modeling, Flexible Pavement, Static Analysis

1. Introduction

Pavement is the layered structure constructed on the natural soil for the movement of people and vehicles. Flexible pavement generally has three layers of asphalt base and sub grade on the natural soil. They have a layered structure with continuous boundary analysis of pavement introduced complexity for simulation of pavement. Researchers have developed various models for the analysis of flexible pavement with various assumptions according to their purpose, accuracy, and efficiency required at the time. These models and methods range from the low level of complexity to very high levels of complexity.

Pavement is a layered structure constructed on the natural soil for the movement of people and vehicles. Flexible pavement in general has three layers of surface, base and sub-grade (Figure 1). The layered structure of the pavement requires continuous boundary analysis which adds complexity in its simulation. Researchers have developed various approaches for the analysis of flexible pavements with various assumptions made according to their purpose

and need. Moreover, the models and methods developed range from those that are simple to those that are highly complex.

To accurately predict the mechanical response of the pavement, researchers over the years have developed various analytical methods. In the 1940s, Burmister [1] and Odemark [2] analyzed flexible pavement responses and designs based on layered elastic theory. They proposed models considering elastic half-space under static loading. Huang and Shah [3], Chen and Huang [4], and Sun [5] modeled pavement as the beams and plates on Winkler type elastic spring foundation. Zaghoul et al.[6] and Lombaert and Degrande [7] modeled pavement as beams and plates on the homogenous or layered half-plane. Yang and Hung [8], Eason [9], Hao and Ang [8] modeled the pavement as a homogenous or layered half-space.

Analytical methods such as Fourier transform, Laplace transform, Fast Fourier Transform (FFT), Fourier series have been used to obtain the response of the pavement with complex boundaries and material properties as done by Eason [9], Baron et

al.[10], Theodorakopoulos et al.[11] etc. As reported by Zaghoul et al.[6], Yang & Hung [12] with the advent of computational techniques and improvement in computational capacities, numerical methods have been widely used for solving complex problems in pavement analysis. Several approaches have been adopted to model pavement using the finite element (FE) models Lu & Wright [13] in their research paper simulated pavement using a two-dimensional 2D plane strain model. Li et al.[14] simulated using 2D FE axisymmetric model to evaluate pavement performance. However, the three-dimensional simulations gave more reasonable results than the two-dimensional simulations when compared with the actual measurements under traffic loading Cho et al.[15]. Zaghoul et al.[6] were among some of the first to develop a 3-D model capable of capturing the response of moving load. Later, several 3-D finite element models have been proposed by various researchers such as Beskou et al.[16], Gungor et al.[17], Huang et al.[18], Yoo & Al-Qadi,[19].

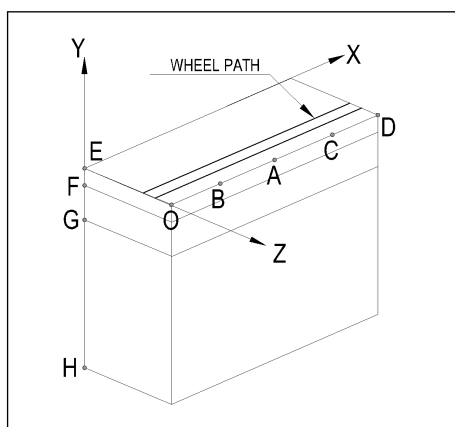


Figure 1: A typical section of flexible pavement

In context of Nepal, Departments of Roads (DoR) [20] provides a guideline for design of flexible pavements which is based on a mechanistic empirical approach. The guideline is based on structural analysis of a multiple layered pavement system using linear elastic model and follows failure criterions similar to that in IRC 37-2001 guideline [21]. There has been some studies on flexible pavements in Nepal but to the knowledge of authors, none of the studies have developed structural analytical models specific to flexible pavements in Nepal. Therefore, this study fills in this gap in literature and is an initial attempt towards development of structural analytical models considering flexible pavements in Nepal.

This study develops a three-dimensional FE model that can be used for structural analysis of flexible pavements in Nepal. The pavement FE model has been simulated in the ANSYS Mechanical APDL [22] and the responses under static loading parameters considering material properties in Nepalese context have been validated. Effect of friction on the pavement responses has been also presented. The developed model can support designers and planners as a pavement analysis tool and can enable them to test flexible pavement related conditions specific to Nepal. The rest of the paper is arranged into four sections. Section 2 details the finite element approach used in the model development. Section 3 discusses the results of numerical tests performed to validate the model and to further analyze the effect of frictional factor and Section 4 concludes the findings of the study.

2. Finite Element Model Generation

A 3D FE model is prepared in ANSYS[23]. Pavement thickness dimensions and materials properties are defined based on recommendations provided by DoR[20]. Following subsections details the structural modeling of the pavement section, material characterization, and the loading mechanisms defined for the analysis.

2.1 Modeling of pavement section and pavement layers

A 3D FE model is created using SOLID185 (8-noded structural solid brick element). SOLID185 is available in two forms in ANSYS, a standard (non-layered) structural solid (KEYOPT (3) = 0) and a layered structural solid (KEYOPT (3) = 1) [23]. The 2x2x2 integration point is taken to calculate stiffness, stress, and mass matrices, while the 2x2-integration point is taken for the pressure load vector.

Flexible pavement generally consists of four layers, surface, base, sub-base, and sub-grade. The elastic material properties of the base and subbase are nearly similar, so only three layers are considered for the pavement's current FE modeling. Therefore, a three layered 3D FE model of the flexible pavement as shown in Figure 2 is created. In the figure, EF, FG and GH represent an asphalt surface, base, and subgrade respectively and the vehicle load moves from point B to C along the wheel path. Overall dimension of the pavement section modeled is 25.8m x 11.5m x 35m.

Figure 2 shows the detailed meshing in the generated FE model. The model has relatively finer mesh, of element size of 0.16 mm along the vehicular path and 0.11 mm in lateral direction near the loading region where high stress is developed. A relatively coarse mesh of average element size of 0.5 mm with their own bias in spacing along its layers is adopted elsewhere. A total of 2,80,323 8-noded solid brick elements are generated for this finite element modeling. The interface between the pavement layers is modeled in two ways: i) perfectly bonded layers having the same displacement between layers and ii) frictional layers allowing the body to slide between layers with the coefficient of friction of 0.15 following.

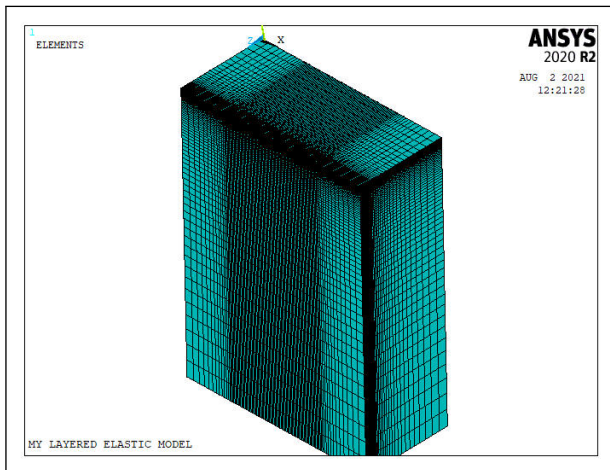


Figure 2: Finite Element Model showing detailed meshing

The boundary condition applied to the pavement structure play an essential role to the pavement response outputs. As pavement is a continuous structure in nature and modeling the pavement with any boundary condition results some error in analysis. Assogba et al.[24], Li et al.[14] have used continuum elements in boundaries to replicate the continuum domain in analysis. Also, absorbing boundaries has been practiced in [16] to minimize the boundary effect on the pavement responses . The analysis in [16] has shown that if a sufficiently large dimension of a model with rollers is considered, the boundary doesn't significantly affect the value of pavement responses as the reaction from boundary gets diminished coming to the point of interest in the model. Having large geometry does provide a more accurate result as boundaries effect on it is minimized, but it does require high computational times. The dimension of the model is determined after conducting various

trials and errors to get a balance between accuracy and computational time. The vertical dimension has a significant effect on the accuracy of the result; thus, a large vertical dimension over other dimensions is considered in the analysis done by Assogba et al.[24], Beskou et al.[16].

Further, the model with roller boundaries provides a similar result with the model having absorbing boundaries [16]. Therefore, this study also utilizes simple roller boundaries in pavement modeling, as illustrated in Figure 3. The vertical displacement decreases as depth increases. It is assumed that the vertical displacement at the bottom of the pavement is negligible; therefore, the bottom of the pavement in the FE model is constrained in the vertical direction. The pavement in nature is restrained in the lateral direction by its surrounding soil, so in the FE model, the lateral movement of pavement is also assumed to be constrained by applying roller on its lateral sides. The vehicle loading and the pavement section subjected to the loading are symmetrical about the x-y plane. Therefore, only half of the pavement and loading is considered during simulation. The roller boundary is applied on the axis of symmetry, and the pavement is restrained sufficiently to prevent rigid body movement under loading.

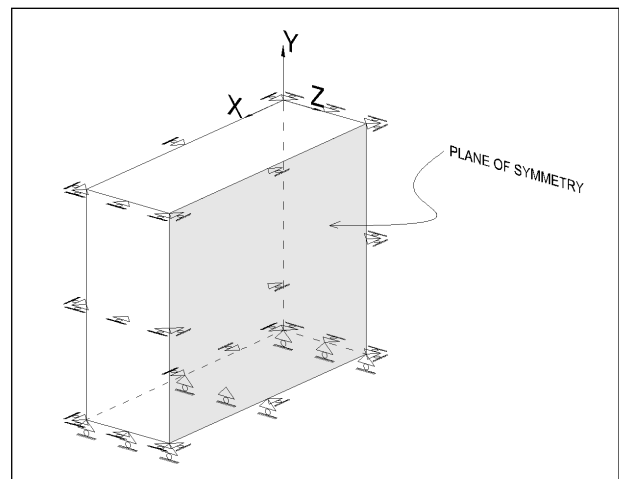


Figure 3: Finite Element Model showing boundary conditions

2.2 Pavement thicknesses and material characterization

The thicknesses and material composition of the pavement layers are taken of commonly used value for a typical flexible pavement in the Kathmandu Valley for design traffic of 13.1 msa (million standard

Table 1: Pavement layer and material composition

Layers	Thickness (m)	Young Modulus (MPa)	Poisson Ratio	Density (kg/m ³)	Material
Asphalt	0.19	2000	0.35	2500	MBS/DBM
Base	0.47	200	0.35	2667	WMM/GSB
Subgrade	34.33	62	0.35	1990	Natural Soil

Note: MBS - Modified Bitumen Surface Course, DBM – Dense bituminous macadam, WMM – Wet mix macadam, GSB – Granular sub base

axle load of 80 kN) satisfying the design recommendations provided by Department of Road (DoR) [20]. Table 1 shows the pavement layer thicknesses, material composition and the elastic material properties of the pavement layers considered for the structural analysis. Sub-grade layer is the natural soil layer and practically extends infinite. To facilitate detailed structural analysis, the total depth of the FE model is defined as 35m that includes a definite large depth of 34.33m of sub-grade layer.

2.3 Modeling of vehicular load

The actual representation of moving vehicle loading in a finite element model is complex. The stresses exerted by vehicles on the pavement are non-uniform and depend on the tire construction, tire load, and tire inflation pressure [25]. To simulate moving vehicle, the stress distribution is considered to be uniform and rectangular, which results in computational simplification. The tire footprint is converted into a rectangle equivalent area (Figure 4) of length and width 0.87L and 0.6L, respectively.

The actual representation of moving vehicle loading in a finite element model is complex. The stresses exerted by vehicles on the pavement are non-uniform and depend on the tire construction, tire load, and tire inflation pressure [25]. The structural analysis in this study is carried out for static vehicular loading considering a uniform stress distribution over an equivalent rectangular contact area, which results in computational simplification. The tire footprint is converted into a rectangle equivalent area as shown in Figure 4 of length and width of 0.87L and 0.6L, respectively. Also, the wheel load varies depending on the axle configuration and the wheel configuration of the vehicle. For numerical modeling purposes and to simplify the APDL code, the tire-pavement contact stress is assumed to be equal to that of inflation pressure of the tire, and the tire-pavement contact surface is assumed to be the same for all tires. Table 2 lists out the values of major loading parameters used

in modeling. The wheel load and the tire pavement contact area are computed based on standard axle load of 80 kN and contact stress of 0.56 MPa recommended by DoR [20] for single axle with dual wheel.

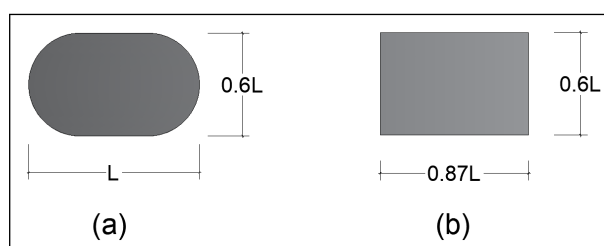


Figure 4: (a) Elliptical (b) Equivalent rectangle footprint of tire

Table 2: Parameters for modeling vehicular load

Parameter	Adopted values
Load On dual wheels (kN)	40
Uniform vertical contact stress (MPa)	0.56
Tire pavement contact area (mm ²)	71428.6
Length of equivalent area (mm)	321.8
Width of equivalent area (mm)	221.9

For FE modeling, each tire footprint’s equivalent length and width are taken as 32 cm and 22 cm, respectively, with the uniform vertical contact stress of 0.57 MPa. The vertical contact pressure was only considered to simulate the dynamic effect of the pavement as the horizontal component is assumed not to have a significant impact on the pavement response according to the principle of Saint-Venant [26]. Further, modeling the three-directional tire-pavement contact forces is beyond the scope of this work.

3. Results and Discussion

The pavement model discussed in the previous section was validated and was then used to simulate and

analyze the pavement responses under the desired load case and material properties for different bonding conditions between the pavement layers. The results of numerical tests are discussed in the following subsections:

3.1 Validation of the FE model

Validation of the modeling, it's meshing, boundary conditions, geometry loading, and the solution is conducted by comparing the result vertical displacement (u_y) and stresses ($\sigma_y, \sigma_x, \sigma_z$) obtained from of the developed model in ANSYS in this study to the theoretical values given in [27] for a single layer pavement system. For the elastic half-space, the theoretical value for the vertical displacement and stresses along the x, y, and z-axis of the pavement under the distributed load, $p = 1.16$ MPa, acting on the circular area of radius α on the surface of the pavement at a depth, y , can be determined by Equation 1, 2 & 3 [27].

$$u_y = \frac{(1 + \nu)p\alpha}{E} \left[\left(1 + \left(\frac{y}{\alpha} \right)^2 \right)^{-0.5} + (1 - 2\nu) \left(1 + \left(\frac{y}{\alpha} \right)^2 \right)^{-0.5} - \left(\frac{y}{\alpha} \right) \right] \quad (1)$$

$$\sigma_y = p \left[1 - \left(1 + \left(\frac{\alpha}{y} \right)^2 \right)^{-1.5} \right] \quad (2)$$

$$\sigma_x = \sigma_z = p \left[\frac{(1 + 2\nu)}{2} - \left(1 + \left(\frac{\alpha}{y} \right)^2 \right)^{-0.5} + 0.5 \left(1 + \left(\frac{\alpha}{y} \right)^2 \right)^{-1.5} \right] \quad (3)$$

Young's modulus of elasticity, E , of 50×10^6 N/m² and Poisson's ratio ν of 0.25 was taken for all three layers as in [16]. The vehicle's load is applied as stationary distributed pressure acting on a rectangular area of 0.46m x 0.3m symmetrical about x-z around surface point A of pavement Figure 2. Table 3 compares the values obtained from the proposed model to the corresponding theoretical values obtained at depth 1m from the surface. The normal

stresses about the X and Z axes are observed to be slightly different, possibly due to unsymmetrical rectangle loading on the surface. The test results shows a reasonable discrepancy of 1-4% between simulated and the theoretical values. The displacement error is observed to be lower than the errors in stresses values. This may be due to the reason that the displacement is a primary result quantity obtained by directly solving the equilibrium equation, while stresses are derived result quantities determined by using primary quantity, so may include some finite element errors.

Table 3: Response of uniform half-space pavement at $z = -1.0$ m due to vertically distributed pressure, $p = 1.16$ MPa acting on a rectangular area of 0.46m x 0.15m on the surface of the pavement

Parameter	Theoretical	Numerical	%Error
u_y (m)	-1.939E-05	-1.901E-05	2.2%
σ_y (kPa)	36.284	35.228	2.91%
σ_x (kPa)	1.477	1.457	1.37%
σ_z (kPa)	1.477	1.538	-4.11%

3.2 Responses of the layered pavement system

After validation of the FE model, the three layered pavement system under the desired loading and material conditions described in Section 2 was tested. For simplicity in modeling, most structural analysis of flexible pavements assumes the pavement layers are perfectly bonded. To investigate the effect of this relaxation, a comparison of the pavement responses for two different contact conditions are presented, namely, a) perfectly bonded model (BONDED MODEL) and b) model with frictional interface (INTERFACE MODEL). The frictional interface was modeled with contact174 and target170 surface element [23] for surface to surface contact pair between asphalt-base and base-sub-grade. The normal frictional stiffness(FKN) was taken to be 0.8 after running various trail to arrive balance between accuracy and computational time as having higher value of FKN result more accurate however is computational demanding.

The pavement responses were determined at critical locations i.e at the asphalt-base ($y = 0.19$ m) and at base-subgrade ($y = 0.67$ m) interfaces. Table 4 compares pavement responses at the two critical interface locations for the interface conditions with and without perfect bonding. Figures 5-8 show the

Table 4: Static Response of Pavement under Vehicular load (DoR, 2021) [20]

Case	y(m)	u _y (mm)	σ _y (kPa)	σ _x (kPa)	σ _z (kPa)	ε _y (x10 ⁶)	ε _x (x10 ⁶)	ε _z (x10 ⁶)
BONDED MODEL	P (y=0.15m)	-0.49	-163	131.1	158.7	-338.4	134.1	104.3
	Q (y=0.67m)	-0.36	-17.26	10.86	9.274	-211	104.3	88
INTERFACE MODEL	P (y=0.15m)	-0.85	-232.6	555.5	612.5	-320.7	211.3	169.2
	Q (y=0.67m)	-0.59	-19.17	38.87	33.58	-222.6	169.2	133.4

displacements and vertical stresses (σ_y) for the two cases which is compression in nature. Test results in Table 4 shows decreases in vertical displacement, stress and strain values of 28 %, 92% and 25%, respectively at the base-subgrade interface than that at the asphalt-base interface for bonded model. Comparison of the cases bonded and interface models shows increments in vertical displacement (u_y) and vertical stress (σ_y) ranging from 38%-42%, and 10%-20%, respectively when the layers are not perfectly bonded.

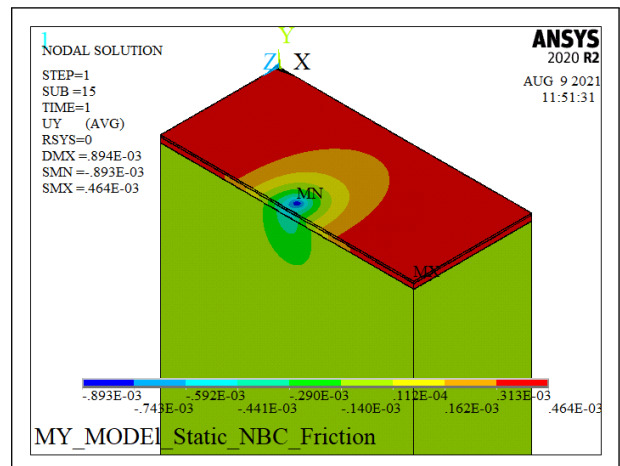


Figure 6: Displacement Response on a friction model due to static vehicular loading

The lateral stresses and strains are tensile in nature and observed to increase by 72% to 76% and 34% to 38% with the relaxation of the perfect bonding condition. With friction contact interface between the layers, the pavement layers can slide which results discontinuities in the pavement responses and this is also reflected in the contour plots of pavement responses in Figures 6 and 8, for the case in which the interfaces are not perfectly bonded.

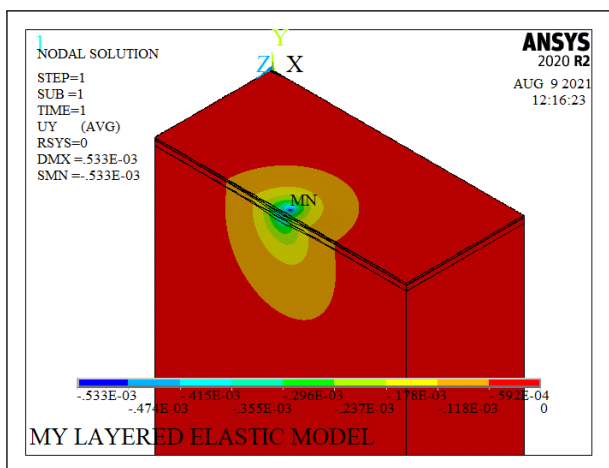


Figure 5: Displacement Response on a bonded model due to static vehicular loading

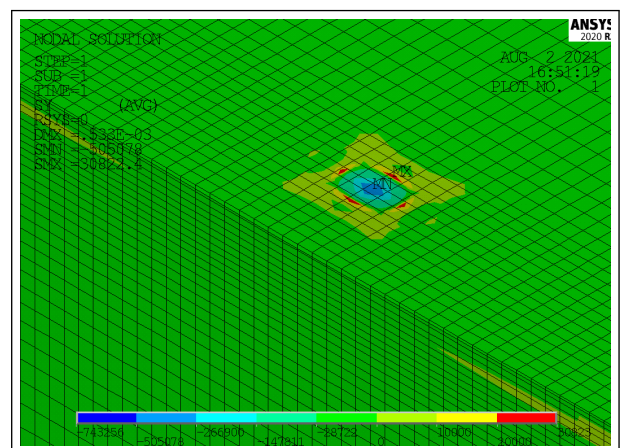


Figure 7: Vertical stress on a bonded model due to static vehicular loading

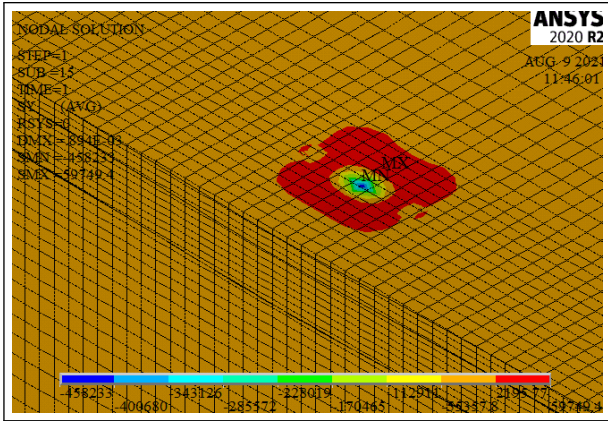


Figure 8: Vertical stress on a friction model due to static vehicular loading

4. Conclusion

A three-dimensional finite Element model suitable for structural analysis of typical flexible pavements in Nepal is developed using finite element software, ANSYS. The vehicular loading and material properties are defined based on DoR, 2014. Following conclusion can be inferred based on the numerical test results obtained from FEM simulation:

- The developed pavement model shows reasonable accuracy in simulating the pavement responses with discrepancy of 1-4% from the theoretical values when compared for the single layered system.
- The pavement responses for typical three layered system with loading and material properties defined in context to Nepal shows decrease in displacement and stress values with increase in depth of the pavement. The vertical displacement, stress and strain values on average decreased by 28% , 92% and 25%, respectively from depth 150 mm to 670 mm from the surface.
- Comparison of the pavement responses for the cases with and without perfectly bonded interfaces between the pavement layers shows the pavement responses, vertical displacement, vertical stress, lateral stress and strains increased on average by 40% 20%, 74%, and 37% with the relaxation of the perfect bonding condition.

Due to time limitations, the present structural analysis of the pavement system does not incorporate the dynamic nature of the vehicular loading and is the

potential area for further research in this direction. Considering the non linear properties of the pavement material and verification with the experimental results are another major areas for future research.

Acknowledgments

The authors acknowledge the support provided by the Center for Infrastructure Development Studies (CIDS), IOE, Tribhuvan University; and Road Board Nepal (RBN) for successful completion of this research study. First author would also like to thank Er Pramod Tiwari, Er Sangita Acharya and Er Prabin Wagle for providing invaluable support during this study.

References

- [1] D. M. Burmister, “The general theory of stresses and displacements in layered soil systems. II,” *Journal of Applied Physics*, 1945.
- [2] S. N. Odemark, *Investigations as to the Elastic Properties of Soils and Design of Pavements According to the Theory of Elasticity*. 1949.
- [3] T. C. Huang and V. N. Shah, “Elastic system ioving on an elastically supported beam,” *Journal of Vibration and Acoustics, Transactions of the ASME*, vol. 106, pp. 292–297, apr 1984.
- [4] Y. H. Chen and Y. H. Huang, “Dynamic stiffness of infinite Timoshenko beam on viscoelastic foundation in moving co-ordinate,” *International Journal for Numerical Methods in Engineering*, vol. 48, pp. 1–18, may 2000.
- [5] L. Sun, “A closed-form solution of beam on viscoelastic subgrade subjected to moving loads,” *Computers and Structures*, vol. 80, pp. 1–8, jan 2002.
- [6] S. Zaghoul, T. White, V. Drnevich, and B. Coree, “Dynamic analysis of FWD loading and pavement response using a three-dimensional dynamic finite-element program,” *ASTM Special Technical Publication*, 1198, pp. 125–125, 1994.
- [7] G. Lombaert and G. Degrande, “Experimental validation of a numerical prediction model for free field traffic induced vibrations by in situ experiments,” *Soil Dynamics and Earthquake Engineering*, vol. 21, pp. 485–497, jul 2001.
- [8] H. Hao and T. C. Ang, “Analytical Modeling of Traffic-Induced Ground Vibrations,” *Journal of Engineering Mechanics*, vol. 124, pp. 921–928, aug 1998.
- [9] G. Eason, “The stresses produced in a semi-infinite solid by a moving surface force,” *International Journal of Engineering Science*, vol. 2, pp. 581–609, mar 1965.
- [10] M. L. Baron, H. H. Bleich, and J. P. Wright, “Ground Shock Due to Rayleigh Waves from Sonic Booms,” *Journal of the Engineering Mechanics Division*, vol. 93, pp. 137–163, oct 1967.

- [11] D. D. Theodorakopoulos, A. P. Chassiakos, and D. E. Beskos, "Dynamic effects of moving load on a poroelastic soil medium by an approximate method," *International Journal of Solids and Structures*, vol. 41, pp. 1801–1822, apr 2004.
- [12] Y. B. Yang and H. H. Hung, "A 2.5D finite/infinite element approach for modelling visco-elastic bodies subjected to moving loads," *International Journal for Numerical Methods in Engineering*, vol. 51, pp. 1317–1336, aug 2001.
- [13] Y. Lu and P. J. Wright, "Numerical approach of visco-elastoplastic analysis for asphalt mixtures," *Computers and Structures*, vol. 69, pp. 139–147, oct 1998.
- [14] M. Li, H. Wang, G. Xu, and P. Xie, "Finite element modeling and parametric analysis of viscoelastic and nonlinear pavement responses under dynamic FWD loading," *Construction and Building Materials*, vol. 141, pp. 23–35, dec 2017.
- [15] Y.-H. Cho, B. F. McCullough, and J. Weissmann, "Considerations on Finite-Element Method Application in Pavement Structural Analysis," *Transportation Research Record: Journal of the Transportation Research Board*, vol. 1539, pp. 96–101, jan 1996.
- [16] N. D. Beskou, G. D. Hatzigeorgiou, and D. D. Theodorakopoulos, "Dynamic inelastic analysis of 3-D flexible pavements under moving vehicles: A unified FEM treatment," *Soil Dynamics and Earthquake Engineering*, vol. 90, pp. 420–431, nov 2016.
- [17] O. E. Gungor, I. L. Al-Qadi, A. Gamez, and J. A. Hernandez, "In-situ validation of three-dimensional pavement finite element models," pp. 145–159, sep 2016.
- [18] B. Huang, L. N. Mohammad, and M. Rasoulian, "Three-dimensional numerical simulation of asphalt pavement at Louisiana accelerated loading facility," vol. 1764, pp. 44–58, jan 2001.
- [19] P. J. Yoo and I. L. Al-Qadi, "Effect of transient dynamic loading on flexible pavements," *Transportation Research Record*, vol. 1990, pp. 129–140, jan 2007.
- [20] DOR, "Pavement Design Guidelines (Flexible Pavement)," tech. rep., 2014.
- [21] K. Koti Marg and R. Puram, "GUIDELINES FOR THE DESIGN OF FLEXIBLE PAVEMENTS INDIAN ROADS CONGRESS," tech. rep., 2018.
- [22] J. A. DeSalva, Gabriel J., Swanson, *ANSYS Mechanical APDL Basic Design Guide*. Houston, Pa. :Swanson Analysis Systems, 1985.
- [23] J. A. DeSalva, Gabriel J., Swanson, *ANSYS engineering analysis system user's manual*. Houston, Pa. :Swanson Analysis Systems, 1985.
- [24] O. C. Assogba, Y. Tan, X. Zhou, C. Zhang, and J. N. Anato, "Numerical investigation of the mechanical response of semi-rigid base asphalt pavement under traffic load and nonlinear temperature gradient effect," *Construction and Building Materials*, vol. 235, p. 117406, feb 2020.
- [25] M. De Beer, "Measurement of tyre/pavement interface stresses under moving wheel loads," *Heavy Vehicle Systems*, vol. 3, no. 1-2, pp. 97–115, 1996.
- [26] B. D. Saint-Venant, "Mémoire sur la torsion des prismes. Mémoires des Savants étrangers," no. 233-560, 1855.
- [27] M. J. Boussinesq, *Application des potentiels*. 1885.