Proceedings of 8th IOE Graduate Conference
Peer Reviewed
ISSN: 2350-8914 (Online), 2350-8906 (Print)
Year: 2020 Month: June Volume: 8

Dynamic Response of Vertical Shaft Pelton Turbine Unit for Free Vibration

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Abstract

An important factor affecting the performance, reliability and life of turbine is mechanical vibration in the turbine unit. Most of the research on vibration of pelton turbine are made for horizontal shaft unit. This work focuses on the vibrational analysis of vertical shaft pelton. The work was carried out to model the equation of motion by calculating kinetic and potential energy using Hamilton's Principle and then using Galerkin method with shape function & Residual function; the angular velocity was determined. Also, the analytically calculated frequency was compared with the simulated result & the dynamic response for free vibration was analyzed.

The pelton wheel is assumed as a shaft disk system; which cover dynamic behavior of vertical flexible shaft, supported by rigid bearing on the other end to determine natural frequency of the system. The critical frequencies were determined for developed mathematical model based on real vertical shaft pelton unit of Kulekhani-I HEP situated in Makawanpur district.

Analytically, Natural frequencies along v and w direction were found to be 53.415 rad/s. The critical speeds along v and w-direction were found to be 510.076 rpm. These analytical results were close to ANSYS simulated result which were found to be 581.05 rpm. Approximated analytical & simulation provided output natural frequency of the unit were not much deviated. The analytical result were compared to simulated results and variation in results were found to be 13.915%.

Keywords

Vertical shaft, Pelton, Natural frequency, Vibration, Campbell diagram, Dynamic behavior

1. Introduction

Pelton turbine is an impulse machine that transforms the potential energy of water into kinetic energy in a form of a water jet, which impacts and drives a Pelton runner. Pelton turbines with either vertical or horizontal shaft are intended for use of water energy at very high head and relatively low discharge. When the natural frequency of the turbine coincides with actual frequency of the turbine causes the formation of the resonance. This resonance forms the increase in chances of failure of the turbine by buckling of deformation of the shaft[1]. The vibration analysis is hence concentrated to the calculation of amplitudes and natural frequency of the system. There is a need to develop methodologies for dynamic analysis of Pelton turbines because prototyping and testing cost are exceptionally high and failure is generally causing great damage in the practical applications and testing of these systems. Moreover, very less work has been done in the field of the dynamic behavior of pelton wheel turbines and their effects in operation and design [1].Research is mostly focused for steam and gas turbines and failure due to effect of sediments and erosion caused. In Nepalese context, some academic research has been made for pelton turbine. But most of these research are based on horizontal shaft unit and has modelled mathematically using lagrange's equation and solved through Rayleigh-ritz method and validation is carried out through calculation of natural frequency,critical speed,torque generation and pressure distributions. Thus, research lacks for the vertical shafted pelton units, this research is made for covering the dynamic behaviour of pelton turbine unit on vertical circular flexible shaft, supported by rigid bearing for the free vibration responses.

1.1 History of Vibration Modelling

The first successful rotor model was proposed by Föppl in 1895. It consisted of a single disk centrally located on a circular shaft, without damping. It demonstrated that supercritical operation was stable. In 1919 Jeffcott conceived the same model, but with damping[2].The model gave rotor behaviours like: critical speed (lateral and torsional), instability conditions and damping.Natural frequencies of a cracked beam are evaluated by representing cracks as massless springs and using a continuous mathematical model of the beam in transverse vibration. Shifrin and Ruotolo utilized this approach to write a determinant equation whose roots are the eigen frequencies of the beam[3].Natural frequencies of rectangular plates are obtained by employing a set of beam characteristic orthogonal polynomials in the Rayleigh-Ritz method[4].A mechanical model is developed of a tapered shaft which is rotating at constant speed about its axis. The effects of shaft tapering and the use of composite materials on the structure's free response are studied. The spatial solutions to the equations of motion are carried out using the general Galerkin method [5].Forced response analysis of an undamped distributed parameter rotating shaft is investigated by using a modal analysis technique. The shaft model includes rotary inertia and gyroscopic effects, and various boundary conditions are allowed. In addition to the modal analysis, Galerkin's method is applied to analyze the forced response of an undamped gyroscopic system. The modal analysis is more complicated, while Galerkin's method is easier to implement and gives an excellent approximation to the closed form solution[6].Bai et al. used ANSYS finite element software to model the main shaft system in the hydro turbine generating unit and calculated the critical speed of rotation [7].D'Alembert principle for shaft in cylindrical co-ordinate system, along with the stress–strain relation, gives the non-homogenous linear differential equation, which can be used to calculate axial stress in the shaft. The axial stress produced by shaft rotation has a major effect on the natural frequency of long high-speed shafts, while shaft diameter has no influence on the results[8].

1.2 Academic Research of Pelton Turbines in Nepal

Rajak et al. researched on dynamic analysis of the Pelton turbine and to obtain the natural frequency of the system, mathematical model developed to calculate the kinetic energy and the strain energy, the equations of motion derived using Lagrange equations and the Rayleigh-Ritz method to study the basic phenomena of cylindrical mode of rotor and validation carried out in Mechanical APDL 14.5 to obtain the critical frequency [9].Research paper by Panthee et al. presented the Computational Fluid Dynamics (CFD) analysis of Pelton turbine of Khimti Hydropower in Nepal. The purpose of CFD analysis is to determine torque generated by the turbine and pressure distributions in bucket for further work on fatigue analysis (Version, 2012).The paper by karki presented the methodologies to study the dynamic response of Pelton turbine unit as a shaft-disk system, mathematical model for dynamic response of the Pelton turbine unit was formulated, and analytical solution of amplitude of forced vibration was found. This methodology can be applied to find the dynamic force response in MHP and other hydropower plants to calculate the acceptance level of vibration analytically and to compare the vibration level during the operation period in long run by measuring the amplitudes using vibration measuring devices [10].The research was carried out in order to perform CFD analysis of Pelton runner of Khimti Hydropower. Whole simulation was performed in ANSYS-CFX. The results obtained from simulation showed high pressure in splitter and deep face of the bucket. The torque calculation was further used to calculate the efficiency and analytical validation of the runner [11].The mathematical models for the real Pelton turbine unit were developed in the form of discrete system models and continuous system model. For discrete system models Foppl/Jeffcot rotor model and Rayleigh's energy method: Effective mass model were used. By calculating the kinetic and strain energy of the shaft and disk, the governing equations of motion were developed for continuous system models. The equations of motion derived by using Lagrange's equation were solved for natural frequencies by Rayleigh-Ritz analytical solution method [12].PhD thesis of Luintel presents the method to study the dynamic response of the shaft of a Pelton turbine. Free vibration analysis of the system is carried out to determine the critical speeds of the system. Mathematical models for the bending vibration of Pelton turbine assembly are developed by assuming Pelton wheel as a rigid disk attached on a flexible shaft. Equations of motions are derived for two different models by modelling the shaft as a rotating Euler-Bernoulli beam and a rotating Timoshenko beam respectively [13].Some unpublished research has been carried for the horizontal simply supported pelton units for master's thesis; pokhrel researched on the dynamic response of horizontal shaft overhung Pelton turbine unit for free vibration and Bhandari researched on the dynamic response of horizontal shaft overhung pelton unit for forced vibration. Similarly, Tharu researched for the vibration analysis of simply supported pelton turbine considering flexible rotor bearing.

2. Development of Mathematical Model

The basic elements considered for developing the mathematical model are the shaft and the disks. The complete mathematical model has been developed in three phases. The equation of motion for the system has been derived using energy method.

The vibration occurring at the real condition is obvious to be non-linear type rather than linear vibration which needs simplification for the development of mathematical model as development of the nonlinear vibration condition is a very complicated task. Thus for simplification, the system to be assumed linear and the disk and bearing to be considered rigid while studying the flexible vertical shaft.

2.1 Total energy of the system

Euler angle

Any rotation can be described by three successive rotations about linearly independent axes and these rotations are Euler angles. The positions, angular velocities and angular accelerations of a body that rotates about a fixed point, such as a gyroscope, and body that rotates about its center of mass (an aircraft, shaft of turbine etc) can be described by Euler's angles: spin ψ, notation θ and precession φ.

Rotational Matrix for 3-1-2 Euler Angles

3-1-2 Euler angle are used to obtain Rotational matrix. The xyz rotated in relation to the XYZ system according to set of angles shown in the figure below. Where general orientation of cross section of the beam element can be obtained by rotation around X axis with angle φ, then by an angle θ_y around new axis y_1 and subsequently by an angle θ_z around z_2.

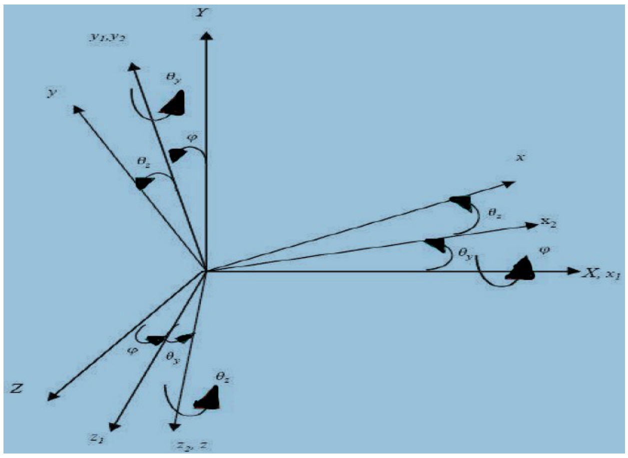

Figure 1: Rotational angles

Angular Velocity of xyz Frame

The instantaneous angular speed ω related to coordinate system XYZ can be seen as:

$$\omega = \dot{\varphi} i + \dot{\theta} y j + \dot{\theta} k \quad (1)$$

Where: i, j and k are the unit vector along the axis x, y_1 and z_2. Transforming Equation (1) for the XYZ coordinate system and assuming the small angles to $\theta_y \theta_z$ will get: [14]

$$\omega = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \dot{\varphi} - \dot{\theta}_z \theta_y \\ \dot{\theta}_y \cos\varphi - \dot{\theta}_z \sin\varphi \\ \dot{\theta}_z \cos\varphi + \dot{\theta}_y \sin\varphi \end{bmatrix} \quad (2)$$

The Shaft

$$T_s = \tfrac{1}{2}\rho_s A_s \int_0^L (V_x^2 + V_y^2 + V_z^2) dx + \tfrac{1}{2} JJJ_p \int_0^L (\dot{\varphi}^2 + 2\dot{\varphi}\dot{\theta}_z \theta_y) dx + \tfrac{1}{2} JJJ_d \int_0^L (\dot{\theta}_y^2 + \dot{\theta}_z^2) dx$$

The strain energy of shaft combine axial deformation and bending can be taken as: [14]

$$U_s = \frac{1}{2}\int_0^L [EA(\frac{du}{dx})^2 + EI(\frac{\partial^2 v}{\partial x^2})^2 + EI(\frac{\partial^2 w}{\partial x^2})^2] dx \quad (3)$$

The Disk

The kinetic energy of disk is given by:

$$T_D = [\frac{1}{2} M_D (v_x^2 + v_y^2 + v_z^2) + \frac{1}{2} I_{pd} (\dot{\varphi}^2 - 2\dot{\varphi}\dot{\theta}_z \theta_y) + \frac{1}{2} I_{dd} (\dot{\theta}_y^2 + \dot{\theta}_z^2)]_{x=L} \quad (4)$$

Total K.E. and P.E.

The total Kinetic Energy of the System is, T = $T_D + T_S$

$$T = [\frac{1}{2}M_D(v_x^2 + v_y^2 + v_z^2) + \frac{1}{2}I_{pd}(\dot{\varphi}^2 - 2\dot{\varphi}\dot{\theta}_z\theta_y) + \frac{1}{2}I_{dd}(\dot{\theta}_y^2 + \dot{\theta}_z^2)]_{x=L} + \frac{1}{2}\rho_s A_s \int_0^L (V_x^2 + V_y^2 + V_z^2)dx + \frac{1}{2}JJJ_p \int_0^L (\dot{\varphi}^2 + 2\dot{\varphi}\dot{\theta}_z\theta_y)dx + \frac{1}{2}JJJ_d \int_0^L (\dot{\theta}_y^2 + \dot{\theta}_z^2)dx \quad (5)$$

The total Potential Energy of the system is,

$$U_s = \frac{1}{2}\int_0^L [EA(\frac{du}{dx})^2 + EI(\frac{\partial^2 v}{\partial x^2})^2 + EI(\frac{\partial^2 w}{\partial x^2})^2]dx \quad (6)$$

Where, ρ denotes mass per unit volume, S is the shaft's cross-sectional area which is assumed constant throughout the length, and I is the second moment of inertia of the shaft cross-section about its neutral axis, ω_x,ω_y and ω_z -the angular velocity, ρ_s – mass per unit volume, A_s – cross-sectional area of shaft assumed to be constant, J_p & J_d - polar moment of inertia & diametral moment of inertia, $M_D$is the mass of the disk and I is the the moment of inertia about the the principal axis, ρ denotes mass per unit volume, S is the shaft's cross-sectional area which is assumed constant throughout the length L.

2.2 Equation of motion of the system

Hamilton's principle states that the motion of the system between a given initial configuration q(t_0) and a given final configuration q(t_1) is such that it extremizes the action integral, [15]

$$I = \int_{t_0}^{t_1} L dt \quad (7)$$

Where "L" is the Lagrange given by L=T-U where T is kinetic energy and U is potential energy of system.[10].

After using Hamilton principle for above system EOM is obtained as below:

$$\rho_s A_s \ddot{u}_1 + M_D \ddot{u}_1 \delta_f(x-L) - EA_s u_1{}'' = 0 \quad (8)$$

$$\rho_s A_s \ddot{v}_1 - \rho_s A_s v_1 \dot{\varphi}^2 + M_D \ddot{v}_1 \delta_f(x-L) - M_D v_1 \dot{\varphi}^2 \delta_f(x-L) + EI v_1{}'''' = 0 \quad (9)$$

$$\rho_s A_s \ddot{w}_1 - \rho_s A_s w_1 \dot{\varphi}^2 + M_D \ddot{w}_1 \delta_f(x-L) - M_D w_1 \dot{\varphi}^2 \delta_f(x-L) + EI w_1{}'''' = 0 \quad (10)$$

$$JJJ_d \ddot{\theta}_{y_1} + JJJ_p \dot{\varphi}\dot{\theta}_{z_1} + JJJ_{dd}\ddot{\theta}_{y_1}\delta_f(x-L) + JJJ_{pd}\ddot{\varphi}\ddot{\theta}_{z_1}\delta_f(x-L) = 0 \quad (11)$$

$$JJJ_d \ddot{\theta}_{z_1} - JJJ_p \dot{\varphi}\dot{\theta}_{y_1} + JJJ_{dd}\ddot{\theta}_{z_1}\delta_f(x-L) - JJJ_{pd}\ddot{\varphi}\ddot{\theta}_{y_1}\delta_f(x-L) = 0 \quad (12)$$

Where, $\delta_f(x-L)$ is Dirac delta function.

2.3 Analytical solution of the system

Galerkin method

For analytical solutions, we used galerkin method. By determining shape function (phi/ϕ) and further calculating residual function (R), analytical solution is obtained.

Galerkein method is expressed as:

$$\int_0^L \phi_1 R dx = 0 \quad (13)$$

Shape function

Assuming the polynomial function as:

$$\phi_1 = C_0 + C_1\bar{x} + C_2\bar{x}^2 + C_3\bar{x}^3 + C_4\bar{x}^4 \quad (14)$$

Where,$\bar{x}$=(non - dimensional) At x = 0 , $\bar{x}$ = 0 and x = L , $\bar{x}$= 1

Calculating $\phi_1{}'$, $\phi_1{}''$ and $\phi_1{}'''$ and applying the BC's;$\phi(0) = \phi'(0) = \phi''(L) = \phi'''(L) = 0$.We get, the value of constants as: $C_0 = 0$, $C_1 = 0$, $C_2 = 6$, $C_3 =$ -4 , $C_4 = 1$ and substituting the values for constants in equation (16), the mode shape for v & w is obtained as $\phi_1 = 6\bar{x}^2 + (-4)\bar{(}x)^3 + \bar{(}x)^4$ i.e.

$$\phi_1 = x^4 - 4Lx^3 + 6L^2x^2 \quad (15)$$

For calculating the mode shape for u, applying the boundary conditions as: u(0) = 0 at x = 0 and $u'(0) = 0$ at x = L. The shape function for first mode of vibration is obtained as

$$\phi_1 = x^2 - 2xL \quad (16)$$

Residual Function
For u, The Residual function for u, v and w is respectively given by following equations:

$$R = \rho_s A_s \ddot{u}_1(t)\phi_1 + M_D \ddot{u}_1(t)\delta_f(x-L)\phi_1 - EA_s u_1(t)\phi_1'' \quad (17)$$

$$R = \rho_s A_s \ddot{v}_1(t)\phi_1 - \rho_s A_s v_1(t)\Omega^2\phi_1 + M_D \ddot{v}_1(t)\delta_f(x-L)\phi_1 - M_D v_1(t)\Omega^2\delta_f(x-L) + EI v_1(t)\phi_1'''' \quad (18)$$

$$R = \rho_s A_s \ddot{w}_1(t)\phi_1 - \rho_s A_s w_1(t)\Omega^2\phi_1 + M_D \ddot{w}_1(t)\delta_f(x-L)\phi_1 - M_D w_1(t)\Omega^2\delta_f(x-L) + EI w_1(t)\phi_1'''' \quad (19)$$

Final Equation of motion
Using the galerkian residual method, the final equation of motions are obtained as:

$$\frac{8}{15}\rho_s A_s L^2 \ddot{u}_1(t) + M_D L \ddot{u}_1(t) + \frac{4}{3} EA_s u_(t) = 0 \quad (20)$$

$$\frac{104}{45}\rho_s A_s \ddot{v}_1(t) L^4 - \frac{104}{45}\rho_s A_s v_1 \Omega^2 L^4 + 9M_D \ddot{v}_1(t) L^3 - 9M_D v_1 \Omega^2 L^3 + \frac{144}{5} EI v_1(t) = 0 \quad (21)$$

$$\frac{104}{45}\rho_s A_s \ddot{w}_1(t) L^4 - \frac{104}{45}\rho_s A_s w_1 \Omega^2 L^4 + 9M_D \ddot{w}_1(t) L^3 - 9M_D w_1 \Omega^2 L^3 + \frac{144}{5} EI w_1(t) = 0 \quad (22)$$

3. Solution of the EOMs

solving the equations (20),(21) and (22), putting the value as, $\Omega = \frac{2\pi N}{60} = 62.832$, $\rho_s = 7850 kg/m^3$, $A_s = \frac{\pi d^2}{4} = \pi * \frac{0.425^2}{4} = 0.14186 m^2$, $L = 3.241m$, $M_D = 3.5ton = 3500kg$. $E = 200Gpa = 200 * 10^9 pa$, $I = \frac{\pi d^4}{64} = \pi * \frac{0.425^4}{64} = 1.6015 * 10^{-3} m^4$.

We get,

$$\ddot{u}_1(t) + 1.52 * 10^7 u_1(t) = 0 \quad (23)$$

$$\ddot{v}_1(t) + 2.8531 * 10^3 v_1(t) = 0 \quad (24)$$

$$\ddot{w}_1(t) + 2.8531 * 10^3 w_1(t) = 0 \quad (25)$$

The above three equation of the motion are in the form of $m\ddot{x} + c\dot{x} + kx = 0$. Thus, natural frequency, $\omega[u_1(t)] = \sqrt{\frac{k}{m}} = \sqrt{\frac{1.52*10^7}{1}} = 3894.708 rad/s$, $\omega[v_1(t)] = \sqrt{\frac{k}{m}} = \sqrt{\frac{2.8531*10^3}{1}} = 53.415 rad/s$ and $\omega[w_1(t)] = \sqrt{\frac{k}{m}} = \sqrt{\frac{2.8531*10^3}{1}} = 53.415 rad/s$.

The solutions of the above equation is obtained as,

$$u_1(t) = A\cos(3894.708)t + B\sin(3894.708)t \quad (26)$$

$$v_1(t) = A\cos(53.415)t + B\sin(53.415)t \quad (27)$$

$$w_1(t) = A\cos(53.415)t + B\sin(53.415)t \quad (28)$$

Now,from the calculated value of angular velocity, the critical speed were calculated using the formula,$\omega = \frac{2\pi N}{60}$ i.e. N= $\frac{\omega*60}{2\pi}$.Hence,Critical speed[N] for u,v and w was calculated to be, $N[u_1(t)]$=37191.72 rpm, $N[v_1(t)]$=510 rpm and $N[w_1(t)]$=510 rpm. The frequency [f] was calculated as $\omega = \frac{2\pi}{T} = 2 * \pi * f$ i.e. f=$\frac{\omega}{2\pi}$ and calculated and found to be $f[u_1(t)]$=619.862 Hz, $f[v_1(t)]$=9.297 Hz and $f[w_1(t)]$=9.297 Hz.

4. Result and analysis

Pelton model
The mathematical model is verified using the data of Pelton turbine of Kulekhani-I HEP and for comparison simulation work is further carried out in ANSYS.
The various technical specifications for Kulekhani –I HPS used for the research purpose are as listed:

PARAMETER	VALUE
Output Power	31000 kw
Rated rpm	600 rpm
Runner diameter tip to tip	2045 mm
Pitch circle diameter	1562 mm
Density of runner material	8050kg/m^3
Youngs modulus of runner material	195 GPa
No of buckets	19
Width of bucket	490 mm
Density of bucket material	8050 kg/m^3
Weight of runner-bucket assembly	3.5 ton
Diameter of shaft	425 mm
Length of shaft	3241 mm
Density of shaft	7850 kg/m^3
Young's modulus of shaft (mild steel)	200 Gpa
Weight of shaft	4.43 ton

Table 1: Technical Specification of K-I HPS

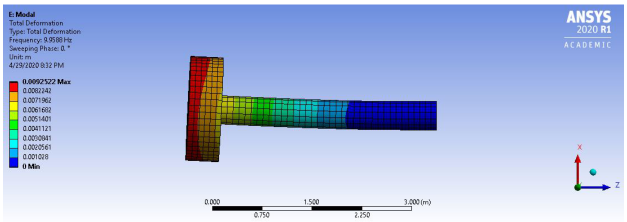

Figure 2: Pelton model of Kulekhani-I HEP at deformation condition at particular spin speed

Analytical Result

The solution for natural frequency of the system is obtained as 53.415 rad/s and 3894.708 rad/s.As the rated speed of the system is 62.83 rad/s at the rated frequency of 50 Hz for maximum torque availability, the Campbell diagram for the mathematical model can be obtained as below:

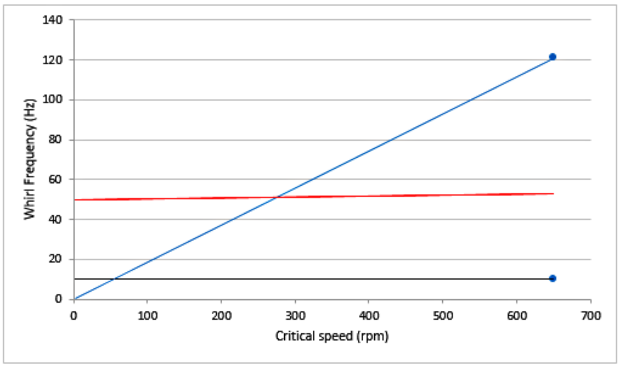

Figure 3: Campbell diagram for Anlytical solution

Simulation result

The model using the similar geometric parameters of the above mentioned Pelton Turbine was made and the corresponding material properties and real constants were fetched in the model. Using the above given data the simulation results, the natural frequencies of the system was found to be 60.848 rad/s and above rated speed of the turbine (i.e. above 62.83 rad/s). The Campbell plot for the assembly was found as shown in figure below:

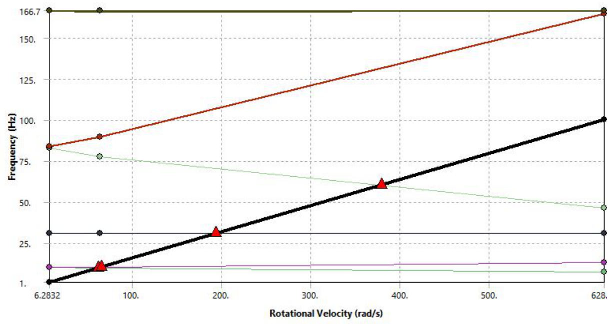

Figure 4: campbell diagram for simulation result

Comparison of Analytical and Simulation Results

Analytically, Natural frequencies along v and w direction were found to be 53.415 rad/s. The critical speeds along v and w-direction were found to be 510.076 rpm. These analytical results were close to ANSYS simulated result which were found to be 581.05 rpm. Approximated analytical & simulation provided output natural frequency of the unit were not much deviated.

The maximum speed that can be reached at the Pelton wheel is called the runaway speed, at which the jet almost completely bypasses the Pelton wheel, without transferring its energy to the rotating buckets. The runaway speed is about twice the nominal speed of the Pelton wheel. Regarding the mechanical safety and the construction costs primarily of the generator and occasionally of the gearbox, the runaway speed is an important factor that must be taken into account already in the design phase of a Pelton turbine. (Runaway Speed and Acceleration Profile, n.d.). Thus, runaway speed of turbine is about 1200 rpm which is far above the rated speed (600 rpm), the analytical result (510.076 rpm) as well as the simulated result for critical speed of (581.05 rpm).

Error percentage calculation between the analytical and simulated result:

The percentage error was calculated to be 13.915% as calculated below:

$$PercentageError = \frac{581.05 - 510.075}{510.075}$$

=0.13915 =13.915%

5. Conclusion

This paper presented the method for vibrational analysis of vertical shaft Pelton Turbine and assembly as a general shaft disk system. The mathematical model for dynamic behavior of the Pelton turbine assembly was thus formulated and the analytical solution of natural frequency and critical speeds was performed. The analytical results for a Pelton turbine setup shows that the natural frequency of the system lies in a good safe range. This also provides the information that the design process followed by the manufacturers is reliable considering the dynamics of the system. The analytical solution and the simulation results were found to be fairly varied from each other. Besides, Rayleigh-Ritz method and Lagranges equation,Hamiltons principle can also be effectively used for developing the mathematical model of vertical shaft pelton unit which has very few research in present context compared to horizontal shafted pelton units.

Also, the sensitivity of various parameters on the dynamic response of the system can be further studied. Different parameters changes such as change in diameter and length of shaft as well as mass of runner-bucket assembly can be studied for the variation of natural frequency of the system.

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