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Modeling and Analysis of Flow Induced Vibration in Pipes Using Finite Element Approach

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Abstract

Flow induced vibration in pipes conveying fluid with different end conditions and different materials is studied in this paper. Four types of end conditions: simply-simply, clamped-clamped, clamped-simply and clamped-free and four materials: aluminum, steel, chlorinated poly-vinyl chloride (CPVC) and concrete are used for the study. Mathematical equation of the flow induced vibration in pipes conveying fluid is developed by using Hamilton's energy principle. Finite Element Analysis is used to study the vibrational characteristics of pipe conveying fluid. The effect of the increase in the fluid velocity on the fundamental frequency of vibration of pipe is studied. Results of the simply-simply supported aluminum pipe are compared with the experimental results for the purpose of validation of finite element model. Natural frequency of vibration and critical flow velocity are determined and vibration characteristics of pipes of different ends conditions and materials are analyzed. Natural frequencies and critical flow velocities from finite element method are compared with the values form direct method. Results indicate that the stability of clamped-clamped supported pipe is very high and the stability of clamped-free supported pipe is very less against flow induced vibration. All the conclusions can be applied in the HVAC pipe installations, petroleum transportation, nuclear installations and other engineering fields for reducing failures due to vibration.

Keywords

Finite element method, flow-induced vibration, natural frequency, critical flow velocity

1. Introduction

Piping systems are widely used in many engineering applications for transferring fluids such as in HVAC, petroleum transportation, municipal water supply, nuclear power plant, hydropower etc. The coupling effect between the fluid and pipe structure can cause pipe vibration and even rupture. Flow induced vibration is undesirable in engineering applications where large amplitude vibrations can cause serious and costly damages and can also put the human life at risk. One of the familiar example is the collapse of Tacoma Narrows Bridge on November 7, 1940 during windstorm. Another most familiar form of the instability is the flailing of an unrestrained garden hose [1]. Therefore, maintaining the stability and reliability of engineering equipment and systems against flow-induced vibration is the challenging problem, which the engineer has to face.

Over the past seven decades, vibration of pipes conveying fluid has been studied extensively. Dodds and Runyana [2] has presented experimentally the effect of high velocity fluid flow on the static and dynamic characteristics of a simply supported pipe in 1965. There is a flow velocity at which the system becomes unstable, which is called critical flow velocity.

An excellent overview is given by M.P. Paidoussis in 1993. The dynamics of the pipes with supported ends, cantilevered pipes or pipes with unusual boundary conditions; continuously flexible pipes, pipes conveying compressible or incompressible fluid, these and many more are the aspects of the problem considered [3]. A great contribution on vibration of pipes conveying fluid was provided by Zang, Gorman and Reese [4]. He derived dynamic equilibrium matrix equation for a discretized pipe element containing flowing fluid from the Lagrange principle, the Ritz method by consideration of coupling between pipe and fluid. He developed a linear vibration model for the vibration analysis of pipes conveying fluid. The model is then used to investigate the vibratory behavior of simply supported pipe subjected to initial axial tensions with three cases: empty pipe, static pipe and pipe conveying fluid. The results from linear vibration model were compared with experimental results.

Subsequently, Kupier and Metrikine [5] conducted a research on stability of clamped-pinned pipe. A tensioned Euler-Bernoulli beam in combination with a plug flow model is used as model. The stability was studied by using D-decomposition method. In this paper analytical proof of stability of a clamped-pinned pipe conveying fluid at a low speed is presented.

Wang and Ni in 2006 [6] studied the stability and chaotic motions of a standing pipe conveying fluid and compared with hanging system developed by Jin [7]. Both studies on hanging as well as standing pipes involves elastic supports and motion limiting constraints producing non-linear force on the pipe as the motion becomes large. From the comparative study, it is shown that the dynamics of the standing pipe is much richer than that of hanging system [6, 7].

Yi-min, Yong-shou, Bao-hui, and Zhu-feng in 2010 [8] used eliminated element-Galerkin method to investigate the natural frequency of fluid structure interaction in pipeline conveying fluid and the natural frequency equations with different boundary conditions are obtained. By considering the Coriolis force, the natural frequency of a straight pipe simply supported at both ends is studied. In the given boundary condition, the four components (mass, stiffness, length and flow velocity) which relate to the natural frequency of pipeline conveying fluid are studied in detail and the results indicate that the effect of Coriolis force on natural frequency is inappreciable [8].

The natural frequency equations of fluid–structure interaction in pipeline conveying fluid with both ends supported is investigated by a direct method [9]. In this article, the direct method is derived from Ferrari's method and used to solve quartic equations. The dynamic equation of pipeline conveying fluid is obtained by Hamilton's variation principle based on Euler–Bernoulli Beam theory. By using the separation of variables method and the derived method from Ferrari's method, the natural frequency equations and the critical flow velocity equations of pipeline conveying fluid with both ends supported are obtained and are compared with the results of natural frequencies obtained by [8] by using eliminated element-Galerkin method.

Shankarachar, Radhakrishna and Babu, 2015 [10] applied Hamilton's principle and variable separable method to derive equation of motion and transcendental frequency equation was derived for linearly restrained end conditions. They considered the pipe as Euler-Bernoulli Beam. Natural frequency of vibration of pipe for three different conditions namely pipe without fluid, pipe with static fluid and pipe with fluid flow are determined. The results obtained from developed model are compared with the results from simulation done by using I-DEAS software and experimental results for validation purpose.

Sutar, Madabhushi and Posa, 2016 [11] derived equation of motion for pipe conveying fluid from energy expressions using Hamilton's Principle. A new transcendental frequency equation is derived for guided end conditions by using separation of variables method to obtain natural frequencies of fluid conveying pipe. Simulation of pipe conveying fluid is modeled by using I-DEAS software and the analysis is done by ABAQUAS software. The results obtained from both analytical and simulation methods are compared for validation purpose and are found to be in good agreement to each other [11].

In this paper, four types of boundary conditions clamped-clamped, simply-simply, clamped-simply and clamped-free are considered and four types of materials steel, CPVC, concrete and aluminum are used because material mechanical properties play and important role for the evaluation of natural frequency and critical flow velocity. Effect of boundary conditions in the dynamic behavior of fluid conveying pipes of four different materials are analyzed and discussed.

2. Mathematical Modeling

Let us consider a fluid conveying pipeline based on the Euler-Bernoulli Beam theory. The pipe shall be supported at both ends by simple support, fixed support or free at one end. The fluid inside the pipe is assumed non-viscous and in-compressible. The mathematical model of the transverse vibration equation of the pipeline conveying fluid is derived by using Hamilton's energy principle. The structure of the pipe is assumed to be small deformation, internal damping and pressurization effects are either absents or neglected. Figure below is the sketch of pipeline conveying fluid with fixed or simply or free end conditions.

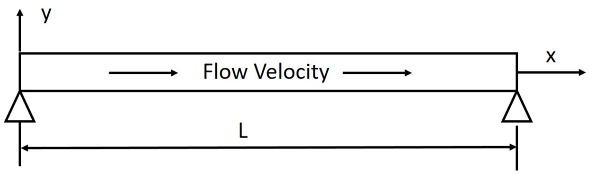

Figure 1: Pipe conveying fluid with supported or free end condition

Equation of motion obtained from Hamilton's energy principle is as follows:

$$EI\frac{\partial^4 y}{\partial x^4}+m_f v^2\frac{\partial^2 y}{\partial x^2}+2m_f v\frac{\partial^2 y}{\partial x \partial t}+M\frac{\partial^4 y}{\partial^4 t}=0 \quad (1)$$

Where, $M=m_p+m_f$. L is length of the pipe, m_p is the mass per unit length of the pipe, m_f is the mass of the fluid per unit length, v is the flow velocity of the fluid, E is the elastic modulus of the pipe material and I is the moment of inertial of the cross section of the pipe.

In the equation (1), first term is the force component acting on the pipe as a result of pipe bending. The second term is the force component acting on the pipe as a result of flow around deflected pipe curvature. The third term is the force required to rotate the fluid element. This force is called Coriolis force. The last term represents the force component acting on the pipe as a result of the inertia of pipe and the fluid flowing through it. The boundary conditions for the pipe with different end conditions are as follows:

Simply-simply support: At x = 0,

$$EI\frac{\partial^2 y}{\partial x^2}=0, \delta y=0$$

At x= L,

$$EI\frac{\partial^2 y}{\partial x^2}=0, \delta y=0$$

Clamped-clampded support: At x = 0,

$$\frac{\partial}{\partial x}\delta y=0, \delta y=0$$

At x= L,

$$\frac{\partial}{\partial x}\delta y=0, \delta y=0$$

Clamped-simply support: At x = 0,

$$\frac{\partial}{\partial x}\delta y=0, \delta y=0$$

At x= L,

$$\frac{\partial}{\partial x}\delta y=0, \delta y=0$$

Clamped-free support: At x = 0,

$$\frac{\partial}{\partial x}\delta y=0, \delta y=0$$

At x= L,

$$EI\frac{\partial^2 y}{\partial^2 x}=0, EI\frac{\partial^3 y}{\partial^3 x}=0$$

3. Natural Frequency and Critical Velocity

The natural frequency equation of beam is given by [12]. When the mass of the beam is replaced by mass of the pipe system, it gives natural frequency of pipe system.

$$w_n=(\beta L)^2\sqrt{\frac{EI}{ML^4}} \quad (2)$$

The value of constant βL for clamped-clamped, simply-simply, clamped-simply and clamped-free conditions are 4.73, 3.142, 3.927 and 1.875 respectively.

The critical load for buckling of beam for different end condition is given by [13]. When this critical buckling load is replaced by load due to the fluid flowing through the pipe, it gives critical velocity of the fluid flowing through the pipe [2].

$$V_{cr}=\frac{c}{L}\sqrt{\frac{EI}{\rho A}} \quad (3)$$

The value of constant c for clamped-clamped, simply-simply, clamped-simply and clamped-free end conditions is 6.285, 3.142, 4.5 and 1.571 respectively.

4. Finite Element Modeling

There are two degrees of freedom (DOFs) at a node in a planner beam elements. They are deflection in the y-direction and the rotation in x-y plane. Hence, each beam element has four DOFs.

To derive the four shape functions, the displacement filed in the element direction can be approximated as follows [12]

$$Y=\sum_{i=1}^{n} N_i a_i \quad (4)$$

Where, N_i are the interpolating shape functions and a_i are a set of unknown parameters. The shape functions N_i are found to be equal to

$$N_1 = \frac{1}{L^3}(2x^3 - 3Lx^2 + L^3) \tag{5}$$

$$N_2 = \frac{1}{L^2}(x^3 - 2Lx^2 + L^2x) \tag{6}$$

$$N_3 = \frac{1}{L^3}(3Lx^2 - 2x^3) \tag{7}$$

$$N_4 = \frac{1}{L^2}(x^3 - Lx^2) \tag{8}$$

Where, L is the length of the pipe element. The potential energy of the pipe element is given by,

$$V = \frac{1}{2}\int_a^b EI(\frac{\partial^2 y}{\partial^2 x})^2\,dx = \frac{1}{2}\int_a^b EI(y")^T(y")\,dx \tag{9}$$

The element stiffness matrix for pipe as beam element is obtained as,

$$[K_1] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

The force acting on the pipe due to the fluid flow is given by,

$$\pi = \int_a^b m_f v^2 \frac{\partial^2 y}{\partial^2 x}\,dx = \int_a^b \rho A v^2 (y')^T(y')\,dx \tag{10}$$

Where, ρ is the density of fluid and A is the cross-sectional area of pipe. The stiffness matrix for the force that conforms fluid to the pipe is obtained as,

$$[K_2] = \frac{\rho A v^2}{30L}\begin{bmatrix} 36 & 3L & -36 & 3L \\ 3L & 4L^2 & -3L & -L^2 \\ -36 & -3L & 36 & -3L \\ 3L & -L^2 & -3L & 4L^2 \end{bmatrix}$$

The stiffness matrix $[K_2]$ tends to weaken the overall stiffness of the pipe system. The force that causes the fluid in the pipe to whip creating instability in the system is represented by dissipation function, which is given by,

$$D = \int_a^b 2m_f v \frac{\partial^2 y}{\partial x \partial t}\,dx = \int_a^b 2\rho A v (y')^T(\dot{y})\,dx \tag{11}$$

The elemental dissipation matrix for Coriolis force is obtained as,

$$[D_1] = \frac{\rho A v}{30}\begin{bmatrix} -30 & -6L & -30 & 6L \\ 6L & 0 & -6L & L^2 \\ 30 & 6L & 30 & -6L \\ -6L & -L^2 & 6L & 0 \end{bmatrix}$$

The kinetic energy of the pipe element is given by,

$$T = M\frac{\partial^2 y}{\partial^2 t} = \int_a^b M(\dot{y})^T(\dot{y})\,dx \tag{12}$$

The elemental mass matrix for the pipe conveying fluid is obtained as,

$$[m_1] = \frac{ML}{420}\begin{bmatrix} 156 & 22L & -54 & 13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}$$

For the formation of global matrices for stiffness, dissipation and mass, first a null matrix of size $[(NumberofNodes \times 2) \times (NumberofNodes \times 2)]$ is formed with its translational and rotational degrees of freedom equal to Number of Nodes. Assembly of elemental stiffness, dissipation and mass matrices to global matrices is performed by MATLAB program. After substitution of boundary conditions global matrices for each support type of pipes are obtained.

5. Dynamic Analysis

The standard equation of motion in the finite element form is given by [14]

$$[M_g](\ddot{y}) + [D_g](\dot{y}) + [K_g](y) = 0 \tag{13}$$

Where, M_g is the global mass matrix, D_g is the global dissipation matrix and $K_g = K_1 - K_2$ is the global stiffness matrix. The above equation has damping term; the solution of eigenvalues problem shall be executed with the characteristics matrix, which is equal to [15]

$$[\Omega] = \begin{bmatrix} -[M_g]^{-1}[K_g] & -[M_g]^{-1}[D_g] \\ I & 0 \end{bmatrix}$$

The solution of the eigenvalue problem gives complex roots. The imaginary part of the roots represents natural frequencies of vibration and the real part represents the rate of decay of the free vibration. The characteristics equation is solved by using MATLAB program.

6. Model Validation

For the validation of the finite element model, the results obtained for simply-simply supported aluminum pipe were compared with the experimental results of Dodds and Runyana [2]. Same parameters as used in experimental analysis were used in finite element analysis too.

Table 1: Parameters for Study

SN	Parameters	Values
1	Length of pipe	3.048 m
2	Outside diameter of pipe	0.0254 m
3	Thickness of pipe	0.00165 m
4	Modulus of elasticity	68.9 GPa
5	Density of pipe	2699 kg/m^3
6	Density of water	1000 kg/m^3
7	Mass of water	0.38 kg/m
8	Total mass of system	0.715 kg/m

Number of elements used for finite element analysis was 50. Results of the Dodds and Runyana [2] experiment and results from the finite element model for simply-simply supported aluminum pipe were as follows:

Table 2: Comparison of Results

Pipe	Flow Velocity, m/sec	Frequency obtained from experiment, rad/sec	Frequency obtained from FE model, rad/sec	Error %
1	0	29.59	30.7853	4.04
	13.10	26.0996	28.9980	11.1
	23.485	24.1116	24.7256	2.54
	29.722	18.8	20.4746	8.9
2	0	not obtained	30.7853	-
	6.59	29.9052	30.34	1.45
	13.973	27.2072	28.7458	5.65
	21.433	26.19	25.8049	-1.45
	29.6826	21.0615	20.5070	-2.63

Results obtained from the finite element model were found to be very close to the experimental result with minimum error. Hence, the model was found to be validated.

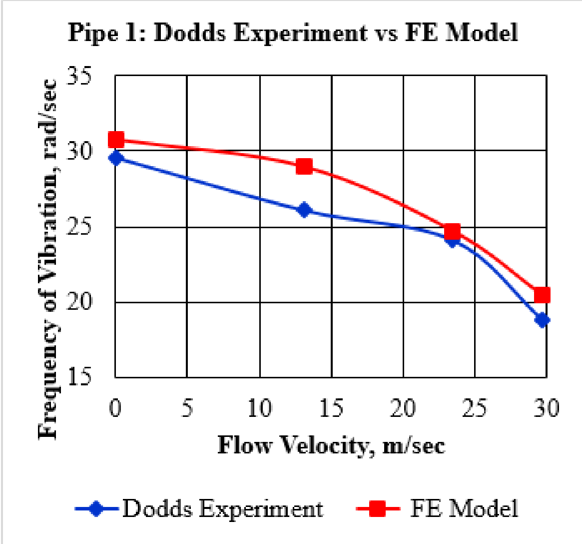

Figure 2: Experimental Results vs Results from Finite Element Model for Pipe 1

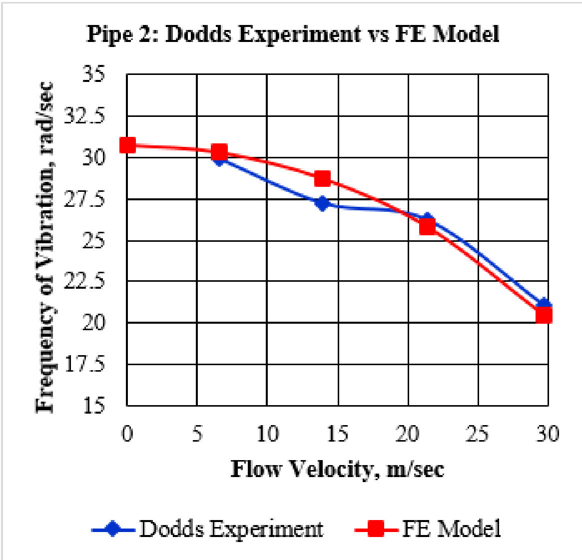

Figure 3: Experimental Results vs Results from Finite Element Model for Pipe 2

7. Results and Discussion

In this section, sixteen different models were analyzed and results were compared. Four materials: steel, CPVC, concrete and aluminum were used for study. Each four materials pipes were analyzed for four types of boundary conditions: simply-simply, clamped-clamped, clamped simply, and clamped-free. Natural frequency of vibration and critical flow velocity were determined for each model. Same parameters used for model validation were used in the analysis too. Young's modulus of elasticity for steel, CPVC and concrete were taken as 207 GPa, 2.9 GPa and 17 GPa respectively. Density of the steel, CPVC and concrete are 8000 kg/m^3, 1550 kg/m^3 and 2400 kg/m^3 respectively. Total mass per unit length of the steel, CPVC and concrete pipe system were 1.386 kg/m, 0.574 kg/m and 0.679 kg/m respectively.

7.1 Aluminum Pipe

Aluminum pipe having the identical size and dimensions was analyzed for four types of boundary conditions. The variation of fundamental natural frequency with the fluid flow velocity is shown in the graph below.

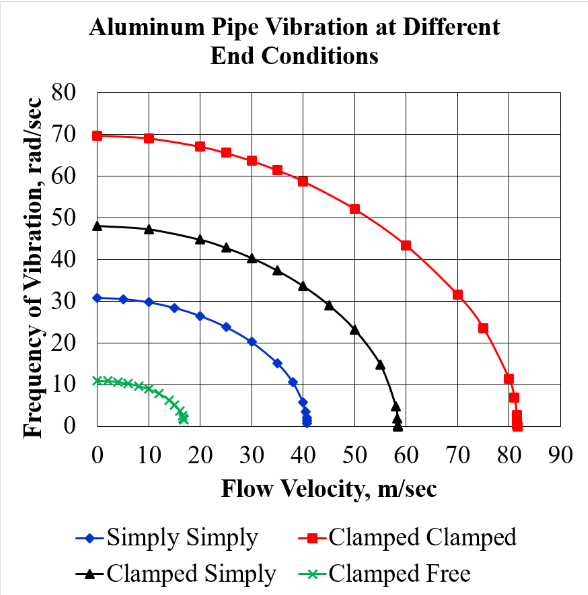

Figure 4: Results Comparison of Aluminum Pipe

The effect of the flowing fluid is to reduce the stiffness and to increase the damping as the flow velocity increases. As a result, frequency of vibration decreases as fluid flow velocity increases. At certain velocity, frequency becomes zero, which is called critical flow velocity. At critical flow velocity pipe becomes unstable. From the above results, it is seen that aluminum pipe shall be stable for large range of flow velocity against flow induced vibration in clamped-clamped condition and shall be stable for small range of flow velocity against flow induced vibration in clamped-free condition. Natural frequency of vibration and critical flow velocity of aluminum pipe from direct method (DM) and finite element method (FEM) at different conditions are as follows.

Table 3: Natural Frequency and Critical Flow Velocity of Aluminum Pipe

End condition	Natural frequency rad/sec		Critical velocity m/sec	
	DM	FEM	DM	FEM
Simply-simply	31.11	30.78	41.25	41
Clamped-clamped	70.51	69.77	82.51	81.60
Clamped-simply	48.60	48.08	59.07	58.36
Clamped-free	11.08	10.96	20.62	16.96

7.2 Steel Pipe

Steel pipe of identical size and dimensions was used in four different boundary conditions. Variation of fundamental frequency of vibration with the fluid flow velocity is shown in graph below.

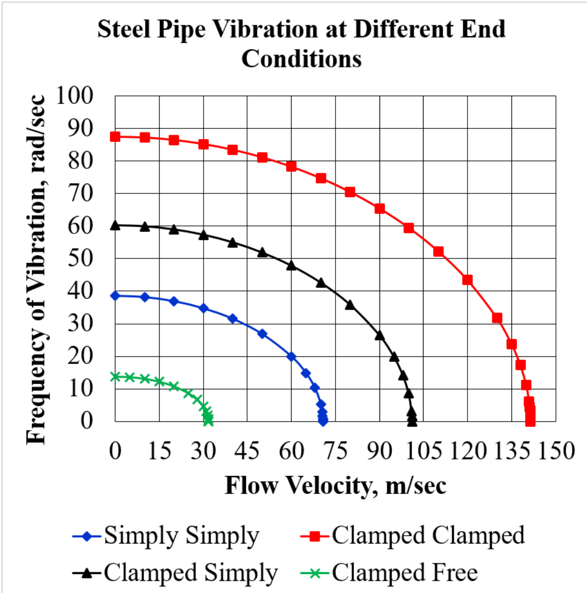

Figure 5: Results Comparison of Steel Pipe

Steel pipe shall be stable for large range of flow velocity against flow induced vibration in clamped-clamped condition and shall be stable for small range of flow velocity against flow induced vibration in clamped-free condition. Natural frequency of vibration and critical flow velocity from direct method (DM) and finite element method (FEM) in each four condition for steel pipe are as follows:

Table 4: Natural Frequency and Critical Flow Velocity of Steel Pipe

End condition	Natural frequency rad/sec		Critical velocity m/sec	
	DM	FEM	DM	FEM
Simply-simply	39.00	38.59	71.49	70.72
Clamped-clamped	88.37	87.47	143.01	141.43
Clamped-simply	60.92	60.28	102.39	101.21
Clamped-free	13.89	13.74	35.75	31.76

Table 5: Natural Frequency and Critical Flow Velocity of CPVC Pipe

End condition	Natural frequency rad/sec		Critical velocity m/sec	
	DM	FEM	DM	FEM
Simply-simply	7.13	7.05	8.46	8.4
Clamped-clamped	16.15	15.98	16.93	16.74
Clamped-simply	11.13	11.01	12.12	12
Clamped-free	2.54	2.2114	4.23	3.36

7.3 Chlorinated Poly-Vinyl Chloride (CPVC) Pipe

Relationship between the fundamental natural frequency of vibration and the fluid flow velocity is shown in the figure below:

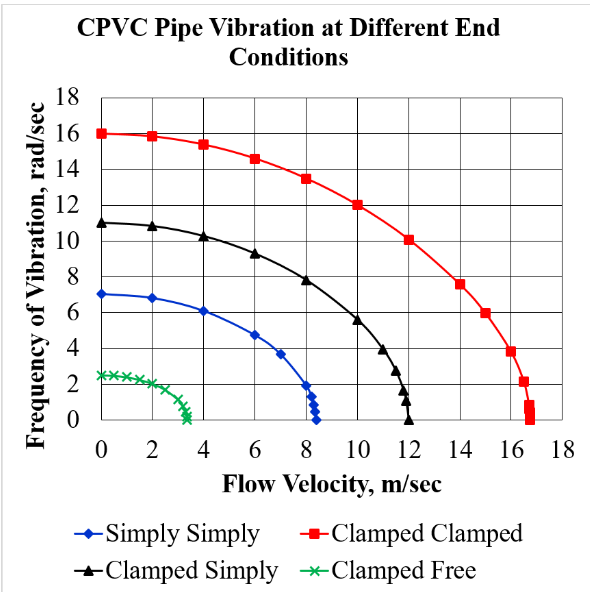

Figure 6: Results Comparison of CPVC Pipe

From the above figure, it is found that CPVC pipe shall be stable for large range of flow velocity in the clamped-clamped condition and shall be stable for small range of flow velocity in the clamped-free condition against flow induced vibration. Natural frequency of vibration and the critical flow velocity from direct method (DM) and finite element method (FEM) in each four types of boundary conditions for CPVC pipe are as shown below:

7.4 Concrete Pipe

Concrete pipe of identical size and dimensions is analyzed in four types of end conditions. Variation of fundamental frequency of vibration with fluid flow velocity is as follows:

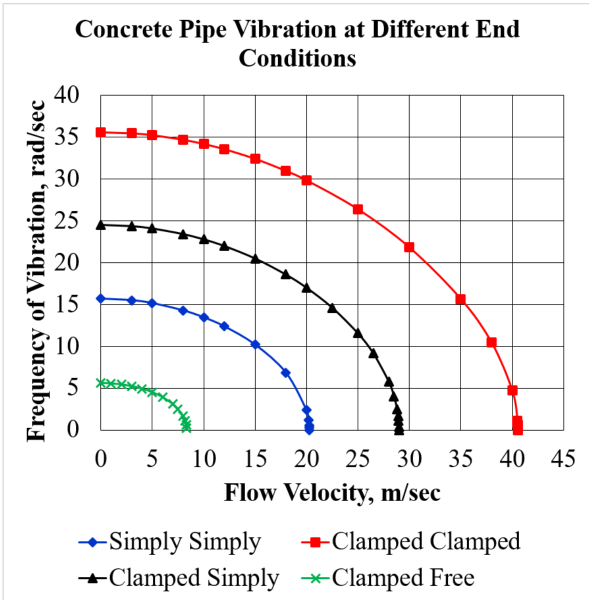

Figure 7: Results Comparison of Concrete Pipe

As in the previously used materials, concrete pipe shall be stable for large range of flow velocity against flow-induced vibration in clamped-clamped condition and it shall be stable for small range of flow velocities against flow-induced vibration in clamped-free condition. Natural frequency of vibration and critical flow velocity from direct method (DM) and finite element method (FEM) in each case of concrete pipe are as follows:

Table 6: Natural Frequency and Critical Flow Velocity of Concrete Pipe

End condition	Natural frequency rad/sec		Critical velocity m/sec	
	DM	FEM	DM	FEM
Simply-simply	15.87	15.70	20.49	20.27
Clamped-clamped	35.96	35.58	40.98	40.53
Clamped-simply	24.79	24.52	29.34	28.99
Clamped-free	5.65	5.59	10.24	8.36

8. Conclusions

The effect of the flowing fluid was to reduce the stiffness and to increase the damping as the flow velocity increases. As a result, fundamental frequency of vibration decreases as fluid flow velocity increases. At certain flow velocity, fundamental natural frequency of vibration was found to be zero. This flow velocity corresponding to zero fundamental frequency of vibration is critical flow velocity. At this flow velocity pipe becomes unstable. Natural frequency of vibration and critical flow velocity were found to be very high for clamped-clamped steel pipe as 87.47 rad/sec and 141.43 m/sec respectively and very low for clamped-free CPVC pipe as 2.2114 rad/sec and 3.36 m/sec respectively. In comparison of the results, it was found that pipes of identical size and dimension but different materials were stable for large range of flow velocities in clamped-clamped conditions and very weak in clamped-free condition against flow induced vibration. The order of the stability with respect to end conditions was found to be clamped-clamped, clamped-simply, simply-simply and clamped-free from higher stability to lower stability. By comparing the results for the pipes having same size and dimensions with respect to the material, steel pipes were found to be stable for large range of flow velocity and CPVC pipes were found to be stable for small range of flow velocities against flow induced vibration. The order of the stability with respect to the material was found to be steel, aluminum, concrete and CPVC from higher stability to lower stability.

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References

[1] Robert D. Blevins. *Flow-Induced Vibration*. Krieger Publishing Company, Malabar, Florida, 2nd edition, 2001.

[2] Harold L. Dodds Jr. and Harry L. Runyan. Effect of high velocity fluid flow on the bending vibrations and static divergence of a simply supported pipe. *National Aeronautics and Space Administration USA*, 1965.

[3] M.P Paidoussis and G.X.L. Pipes conveying fluid: A model dynamical problem. *Journal of Fluids and Structure*, 7:137–204, 1993.

[4] YL Zhang, DG Gorman, and JM Reese. Analysis of the vibration of pipes conveying fluid. *Institution of Mechanical Engineers*, 213(C):849–860, 1999.

[5] G.L. Kupier and A.V. Metrikine. On stability of a clamped-pinned pipe conveying fluid. *HERON*, 49(C):211–232, 2004.

[6] L. Wang and Q. Ni. A note on the stability and chaotic motions of a restrained pipe conveying fluid. *Journal of Sound and Vibration*, 296:1079–1083, 2006.

[7] J.D. JIN. Stability and chaotic motions of a restrained pipe conveying fluid. *Journal of Sound and Vibration*, 208(3):427–439, 1997.

[8] Huang Yi-min, Liu Yong-Shou, Li Bao-hui, Li Yan-jiang, and Yue Zhu-feng. Natural frequency analysis of fluid conveying pipeline with different boundary conditions. *ELSEVIER: Journal of Nuclear Engineering and Design*, 240:461–467, 2010.

[9] Huang Yi-min, Ge Seng, Wu Wei, and He jie. A direct method of natural frequency analysis on pipeline conveying fluid with both end supported. *ELSEVIER: Journal of Nuclear Engineering and Design*, 253:12–22, 2012.

[10] S. M. Shankarachar, M. Radhakrishna, and P. Ramesh Babu. An experimental study of flow induced vibration of elastically restrained pipe conveying fluid. *Proceedings of the 15th International Mechanical Engineering Congress & Exposition*, 2015.

[11] Shankarachar Sutar, Radhakrishna Madabhushi, and Ramesh Babu Poosa. Finite element analysis of piping vibration with guided supports. *International Journal of Mechanical Engineering and Automation*, 3(3):96–106, 2016.

[12] S. S. Rao. *Mechanical Vibrations*. Pearson Education, New York, 5th edition, 2010.

[13] Ferdinand P. Beer, E. Russel Johnston Jr., John T. DeWolf, and David F. Mazurek. *Mechanics of Materials*. McGraw Hill, New York, 6th edition, 2012.

[14] Leonard Meirovitch. *Fundamentals of Vibrations*. Mc Graw Hill, New York, 2001.

[15] Leonard Meirovitch. *Computational Methods in Structural Dynamics*. Sijthoff and Noordhoff, Maryland, USA, 1980.